Question (1978 STEP I Q12)

A sequence of numbers $u_1, u_2, u_3 \dots$ is defined by the relations

$$u_1 = a + b$$
$$u_n = a + b - \frac{ab}{u_{n-1}},$$

where $a + b \neq 0$. Show that if $a \neq b$ then

$$u_n = \frac{a^{n+1} - b^{n+1}}{a^n - b^n},$$

and when $a > b > 0$ determine the limit to which u_n tends as n tends to infinity. Find a formula for u_n when $a = b$, and determine the limit to which u_n tends as n tends to infinity.

None

Question (1969 STEP II Q6)

Let the sequence (x_n) of positive numbers be defined by

$$(1) \quad x_1 = 6, \quad \text{and} \quad (2) \quad x_{n+1} = \sqrt{8x_n - 15}.$$

Show that $5 < x_{n+1} < x_n$ for all n, and that $x_n \to 5$ as $n \to \infty$. Discuss what happens when (1) is replaced by $x_1 = 4$.

None

Question (1972 STEP II Q12)

A sequence of functions $P_n(x)$, $n = 0, 1, 2, \dots$, is defined by setting

$$P_0(x) = 1, \quad P_1(x) = x,$$
$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

and requiring

$$P_n(x) = \sum_{r=0}^{n} A(n,r)x^r.$$

if $n \geq 2$. Show that $P_n(x)$ is a polynomial of degree n, say Construct a flow diagram for the evaluation of the coefficients in $P_N(x)$ for a given value of $N \geq 2$.

None

Question (1982 STEP II Q5)

Let a and b be real numbers with $a > 0$. Successive terms in the sequence $\{f_n\}$ of real numbers are related by

$$f_{n+1} = af_n + b$$

(i) If r is any real root of the polynomial $x^3 - ax - b$, prove that $f_n - r$ has the same sign for all values of n.

(ii) Now suppose that $x^3 - ax - b$ has three real roots r_1, r_2, r_3 with $r_1 < r_2 < r_3$. Prove that $\{f_n\}$ is an increasing sequence if $f_1 < r_1$ or $r_2 < f_1 < r_3$ but is decreasing or constant otherwise.

None

Question (1983 STEP II Q11)

The "logistic" difference equation is

$$x_{n+1} = ax_n(1 - x_n),$$

where $1 < a < 4$. Show that if either $x_1 < 0$ or $x_1 > 1$, then $x_n \to -\infty$ as $n \to \infty$, but if $0 < x_1 < 1$, then $0 < x_n < 1$ for all n. Show further that if x_n tends to a finite limit x as $n \to \infty$, then $x = 0$ or $x = 1 - 1/a$. By writing $x_n = x + \epsilon_n$, and considering $\epsilon_{n+1}/\epsilon_n$, or otherwise, show that sequences x_n with x_1 sufficiently close to $1 - 1/a$ get steadily closer to $1 - 1/a$ provided $a < 3$.

None

Question (1974 STEP III Q2)

Let u_1 be an odd positive integer greater than 1. For $n > 1$, u_n is defined by the relation

$$u_n = u_{n-1}^2 - 2.$$

Show that, for $n > 1$, 1 is the highest common factor of u_n and u_m for $1 \leq m \leq n-1$. Show further that 1 is the highest common factor of u_n and $u_m - 1$ for $1 \leq m \leq n$.

None

Question (1959 STEP II Q102)

Two numbers a and b are given such that $a > b > 0$. Two sequences a_n and b_n $(n = 0, 1, 2, \ldots)$ are defined by the rules:

(i) $a_0 = a$, $b_0 = b$;

(ii) a_{n+1} is the arithmetic mean, and b_{n+1} is the harmonic mean, between a_n and b_n, i.e.
$$2a_{n+1} = a_n + b_n, \quad \frac{2}{b_{n+1}} = \frac{1}{a_n} + \frac{1}{b_n} \quad (n \geq 0);$$

and d_n is defined by
$$d_n = \frac{a_n - b_n}{a_n + b_n}.$$

Prove that $b_{n+1} < a_n^2$ $(n \geq 0)$. Prove that $a_n \to g$ and $b_n \to g$ as $n \to \infty$, where g is the geometric mean between a and b.

None

Question (1961 STEP II Q109)

(i) Find
$$\lim_{n\to\infty} \{\sqrt{n^2 + n + 1} - n\}.$$

(ii) Positive numbers x_0 and y_0 are given. x_1 and y_1 are the arithmetic and geometric means of x_0 and y_0; x_2 and y_2 are the arithmetic and geometric means of x_1 and y_1; and so on. Show that x_n and y_n tend to finite limits as n tends to infinity, and that these limits are equal.

None

Question (1964 STEP II Q105)

A set of functions $y_n(x)$, $(n = 0, 1, 2, \ldots)$ is defined by
$$y_n(x) = \cos(n \cos^{-1} x).$$

Show that

(a) $y_{n+1} - 2xy_n + y_{n-1} = 0$;

(b) y_n is a polynomial in x of degree n;

(c) y_n satisfies the differential equation
$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0;$$

(d) the roots of the equation $y_n(x) = 0$ all lie in the range $-1 < x < 1$.

None

Question (1956 STEP II Q406)

The series of polynomials $f_n(x)$ for $n = 0, 1, 2, \ldots$ are defined by

$$f_n(x) = x^{2n+2}e^{1/x}(d/dx)^{n+1}e^{-1/x}.$$

Prove that for $n \geq 1$

$$f_n(x) = -(2nx - 1)f_{n-1}(x) + x^2 f'_{n-1}(x).$$

Hence show by induction that $f_n(x)$ is a polynomial in x of degree n, and find the coefficient of the highest term. Prove further that the equation $f_n(x) = 0$ has n distinct real roots.

Question (1946 STEP IIII Q305)

A sequence of non-negative numbers $u_0, u_1, u_2, \ldots$ is defined by the recurrence relations

$$u_n^2 = 3u_{n-1} - 2 \quad (n \geq 1)$$

in terms of the first member of the sequence u_0. It is given that $1 < u_0 \leq 2$. Show that $u_n \geq u_0$ and that

$$0 \leq 4 - u_n^2 \leq \left(\frac{3}{2 + u_0}\right)^n (4 - u_0^2)$$

for all $n \geq 0$. Hence, or otherwise, prove that u_n tends to a definite limit as n tends to infinity and evaluate this limit for each u_0.

Question (1923 STEP IIII Q304)

Explain briefly the theory of recurring series, shewing that if $2r$ terms of the series are given it can in general be continued as a recurring series of the rth order in one way only. Find the $(n + 1)$th term of the recurring series

$$-2 + 2x + 14x^2 + 50x^3 + \ldots.$$

Question (1934 STEP III Q508)

The solid angle subtended at a point O by a plane area may be defined as the area cut off on a sphere of unit radius whose centre is O by the straight lines joining O to the perimeter of the plane area. Find the solid angle subtended by a circle at a point on the line through its centre and perpendicular to its plane in terms of α, the angle subtended at the point by a radius of the circle.

Shew also that a rectangle of sides $2a, 2b$ subtends a solid angle

$$4\sin^{-1}\frac{ab}{\sqrt{(a^2 + h^2)(b^2 + h^2)}}$$

at a point on the line through its centre and perpendicular to its plane, where h is the perpendicular distance of the point from the plane.