Question (1970 STEP I Q6)

Show that the geometric mean of n positive numbers is less than or equal to their arithmetic mean. Use this result and the binomial theorem to show that

$$(n+1)! \leq 2^n \left[1! \cdot 2! \cdots n!\right]^{2/(n+1)}$$

$$\begin{aligned}
2^n &= \sum_{k=0}^{n} \binom{n}{k} \\
&\geq (n+1) \sqrt[n+1]{\prod_{k=0}^{n} \binom{n}{k}} \\
&= (n+1) \sqrt[n+1]{\frac{n!}{0! \cdot n!} \cdot \frac{n!}{1! \cdot (n-1)!} \cdots \frac{n!}{n! \cdot 0!}} \\
&= (n+1)^{n+1} \sqrt[n+1]{\frac{(n!)^{n+1}}{(0! \cdot 1! \cdots n!)^2}} \\
&= \frac{(n+1)!}{\left[1! \cdot 2! \cdots n!\right]^{2/(n+1)}}
\end{aligned}$$

$$\Rightarrow \qquad (n+1)! \leq 2^n \left[1! \cdot 2! \cdots n!\right]^{2/(n+1)}$$

Question (1972 STEP I Q5)

By writing $n^{1/n} = 1 + x_n$ and using the fact that $(1+x)^n \geq \frac{1}{2}n(n-1)x^2$ if $n \geq 2$ and $x \geq 0$, prove that $n^{1/n}$ tends to 1 as $n \to \infty$. Sketch the graph of $y = x^{1/x}$ for $x > 0$.

Question (1972 STEP I Q13)

Let n be a positive integer, and consider the sequence $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$, where $\binom{n}{r}$ denotes the binomial coefficient $\frac{n!}{r!(n-r)!}$. (i) Show that no three consecutive terms of the sequence can be in geometric progression. (ii) Show that if there are three consecutive terms $\binom{n}{r-1}, \binom{n}{r}, \binom{n}{r+1}$ in arithmetic progression, then $(n-2r)^2 = n+2$, and find an n for which there are three such terms. (iii) Show that it is never possible to have four consecutive terms of the sequence in arithmetic progression.

Question (1973 STEP I Q1)

(i) Show that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$ for each positive integer n, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. (ii) Show that, for all positive integers r and n, $\sum_{s=0}^{n} \binom{r+s}{r} = \binom{n+r+1}{r+1}$.

Question (1973 STEP I Q3)

Let a be a positive integer, and write $r = \sqrt{a} + \sqrt{(a+1)}$. Show, for each positive integer n, that $a_n = \frac{1}{4}(r^{2n} + r^{-2n} - 2)$ is an integer, and that $r^n = \sqrt{a_n} + \sqrt{(a_n+1)}$. [Positive square roots are to be taken throughout.]

Question (1974 STEP I Q2)

Prove that, if $0 \leq r \leq n$, then $\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$. Hence or otherwise show that, for $n \geq 4$,

$$\sum_{i=0}^{n} i^4 = 24\binom{n+1}{5} + 36\binom{n+1}{4} + 14\binom{n+1}{3} + \binom{n+1}{2}$$

(The binomial coefficient $\binom{n}{r}$ is defined by $\binom{n}{r} = \frac{n!}{r!(n-r)!}$; by convention, $0! = 1$.)

Question (1974 STEP I Q16)

Show by using the binomial expansion or otherwise that $(1+x)^n \geq nx$ whenever $x \geq 0$ and n is a positive integer. Deduce that if $y > 1$ then, given any number K, we can find an N such that $y^n \geq K$ for all integers $n \geq N$. Show similarly that if $y > 1$ then, given any K, we can find an N such that $\frac{y^n}{n} \geq K$ for all integers $n \geq N$.

Question (1975 STEP I Q7)

State precisely, without proof, the arithmetic-geometric mean inequality. The equation $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = 0$ has n distinct positive roots. Writing $a_i = (-1)^i\binom{n}{i}b_i$, where $\binom{n}{i}$ denotes the usual binomial coefficient, prove that $b_{n-1} > b_n$. By considering $f'(x)$, or otherwise, prove further that $b_1 > b_2 > ... > b_{n-1} > b_n$.

Question (1976 STEP I Q1)

Prove that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$. Hence prove that for $n > r$ $\binom{n}{r} = \sum_{i=0}^{r}\binom{n-i-1}{r-i}$. [The binomial coefficient $\binom{n}{r}$ is defined by $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$.]

Question (1977 STEP I Q9)

Show that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

and hence, by induction or otherwise, evaluate

$$\sum_{q=0}^{n}\binom{n+q}{q}\frac{1}{2^{n+q}}.$$

[The binomial coefficient $\binom{n}{r}$ is defined by $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.]

Question (1978 STEP I Q5)

By considering $(1-1)^n$, prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0,$$

for $n = 1, 2, \ldots$. Hence or otherwise prove by induction that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = \binom{n}{1}\frac{1}{1} - \binom{n}{2}\frac{1}{2} + \ldots + (-1)^{n-1}\binom{n}{n}\frac{1}{n}$$

for $n = 1, 2, \ldots$. [You may assume without proof that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$]

Question (1984 STEP I Q3)

By looking at the coefficient of x^n in $(1+x)^{2n}$ in two different ways, or otherwise, show that

$$\sum_{r=0}^{n} \left(\frac{1}{r!(n-r)!} \right)^2 = \frac{(2n)!}{(n!)^2}.$$

By applying the theorem of the arithmetic and geometric means deduce that

$$\left(\frac{(n!)^2(n+1)}{(2n)!} \right)^{(n+1)/4} \leq 1!2!3! \ldots n!.$$

Question (1974 STEP III Q1)

Prove the Binomial Theorem, that

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

(where n is a positive integer). Let m, k be positive integers. By considering the powers of 2 occurring in the numerator and denominator, or otherwise, show that $\binom{2^k m}{2^k}$ and m are either both even or both odd. Deduce that if the coefficients of x^r in $(1+x)^n$ are all odd, then $n + 1 = 2^k$ for some k.

Question (1975 STEP III Q3)

Given a sequence $u_0, u_1, u_2, \ldots$ we define a new sequence $u'_0, u'_1, u'_2, \ldots$ by

$$u'_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} u_{n-i},$$

where $\binom{n}{i}$ denotes the usual binomial coefficient. Applying the same process to the sequence $u'_0, u'_1, u'_2, \ldots$ we obtain a sequence $u''_0, u''_1, u''_2, \ldots$. Show that $u''_n = \sum_{i=0}^{n} c_{n,i} u_i$ for some coefficients $c_{n,i}$, and find these coefficients. If the sequence $u_0, u_1, u_2, \ldots$ satisfies the recurrence relation $u_n = n u_{n-1}$, show that $u'_0, u'_1, u'_2, \ldots$ satisfies $u'_n = n u'_{n-1} + (-1)^n u_0$.

Question (1980 STEP III Q1)

Let k, n be integers, $k \geq 1$, $n \geq 1$. Show that if n^2 divides $(n+1)^k - 1$ then n divides k, and deduce that if $(n+1)^k - 1 = n!$ and $n \geq 6$, then n divides k. [Hint: is n odd or even?] Hence find all pairs (n, k) of positive integers such that $(n+1)^k - 1 = n!$

Question (1965 STEP I Q3)

A monomial of degree n in the m variables $x_1, x_2, \ldots, x_m$ is defined to be an expression of the form

$$x_1^{t_1} \ldots x_m^{t_m}$$

where each of $t_1, \ldots, t_m$ is a non-negative integer and $t_1 + \ldots + t_m = n$. Find the number of monomials of degree n in m variables, and show that the number of monomials of degree $\leq n$ in m variables is

$$\frac{(m+n)!}{m!\, n!}.$$

Question (1961 STEP I Q103)

Writing $C(n,r)$ for $\frac{n!}{r!(n-r)!}$ (and taking $C(n,0) = C(n,n) = 1$), prove that, if $0 \leq r \leq n$,

$$\sum_{r=0}^{s} (-1)^r C(n,r) C(n, s-r) = 0 \quad \text{if } s \text{ is odd}$$

$$= (-1)^{s/2} C(n, \frac{s}{2}) \quad \text{if } s \text{ is even.}$$

What can you say about

$$\sum_{r=0}^{s} (-1)^r C(n+r-1, r) C(n+s-r-1, s-r)?$$

Justify your answer.

Question (1959 STEP IIII Q204)

For each positive integer n, let

$$u_n = 1 - (n-1) + \frac{(n-2)(n-3)}{2!} - \frac{(n-3)(n-4)(n-5)}{3!} + \cdots$$

where the summation stops with the first term that is equal to 0. By considering $u_{n-1} - u_n$ or otherwise, prove that u_n satisfies a recurrence relation of the form

$$au_n + bu_{n+1} + cu_{n+2} = 0,$$

and determine the relation. Hence, or otherwise, evaluate u_n for general n; in particular, show that $u_n = 0$ whenever $n-2$ is a multiple of 3.

Question (1959 STEP IIII Q303)

If $(1+x)^n = a_0 + a_1x + \ldots + a_nx^n$, find

(i) $\sum_{r=0}^{r=n} \frac{(-1)^r a_r}{r+1}$,

(ii) $\sum_{r=0}^{r=n-k} a_r a_{r+k}$.

If further, n is of the form $4m+2$, find

$$a_1 - a_3 + a_5 - \ldots + a_{n-1}.$$

Question (1962 STEP IIII Q304)

If $a_r = r!(n-r)!$ for $0 < r < n$ and $a_0 = a_n = n!$, prove that $\frac{1}{a_0^2} + \frac{1}{a_1^2} + \cdots + \frac{1}{a_n^2} = \frac{(2n)!}{(n!)^4}$.

Question (1960 STEP II Q403)

The numbers c_0, c_1, $\ldots$, c_n are defined by the identity

$$(1+x)^n = c_0 + c_1x + \ldots + c_nx^n;$$

prove that $\sum c_ic_j$ summed over all integer pairs i, j such that $0 \le i < j \le n$ is equal to

$$2^{2n-1} - \frac{(2n-1)!}{n!(n-1)!}.$$

Question (1963 STEP II Q304)

Prove that the binomial coefficient $\binom{a+b}{b}$ is odd if and only if, when a and b are expressed in binary notation and added, there is no 'carrying over'.

Question (1952 STEP I Q104)

Prove that, if $(1+x)^n = c_0 + c_1 x + \cdots + c_n x^n$, then

(i) $\dfrac{c_0}{1} - \dfrac{c_1}{2} + \dfrac{c_2}{3} - \cdots + (-1)^n \dfrac{c_n}{n+1} = \dfrac{1}{n+1}$;

(ii) $c_0^2 - c_1^2 + c_2^2 - \cdots + (-1)^n c_n^2$ is equal to $(-1)^m (2m)!/(m!)^2$ if n is an even integer $2m$. Find its value when n is an odd integer.

Question (1956 STEP IIII Q203)

Prove that, if the roots of the equation

$$x^n - \binom{n}{1} p_1 x^{n-1} + \cdots + (-)^r \binom{n}{r} p_r x^{n-r} + \cdots + (-)^n p_n = 0,$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $p_n \neq 0$, are all real and positive, so are the roots of the equation

$$x^{n-1} - \binom{n-1}{1} p_1 x^{n-2} + \cdots + (-)^r \binom{n-1}{r} p_r x^{n-r-1} + \cdots + (-)^{n-1} p_{n-1} = 0.$$

Deduce that

$$p_{r-1} p_{r+1} < p_r^2 \quad (1 < r < n; p_0 = 1),$$

stating when equality occurs. Prove also that

$$p_1 \geq p_2^{1/2} \geq p_3^{1/3} \geq \cdots \geq p_n^{1/n}.$$

None

Question (1952 STEP IIII Q302)

Prove that, if n is a positive integer, $(1+x)^n$ can be expressed in the form

$$c_0 + c_1 x + \cdots + c_n x^n,$$

where c_r depends only on n and r, and find the value of c_r. Find the sums of the series

$$\text{(i)} \sum_{r=0}^{r=n-k} c_r c_{r+k}; \quad \text{(ii)} \sum_{r=0}^{r=n} \frac{c_r}{(r+1)(r+2)}.$$

Question (1955 STEP IIII Q304)

If $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, prove that

$$\sum_{n=1}^{N} \binom{n+r-1}{r} = \binom{N+r}{r+1}.$$

Hence express $\sum_{n=1}^{N} n^4$ as a polynomial in N.

Question (1951 STEP II Q305)

Let m be a positive integer and $y \neq \pm 1$. Put

$$(m,0) = 1; \quad (m,j) = \frac{(1-y^{2m})(1-y^{2m-2})\dots(1-y^{2m+2-2j})}{(1-y^2)(1-y^4)\dots(1-y^{2j})} \quad (0 < j \leq m).$$

Show that, for $0 \leq j < m$,

$$(m+1, j+1) = (m, j+1) + y^{2m-2j}(m,j),$$

and hence, or otherwise, that

$$(m,0)+(m,1)y+(m,2)y^2+\dots+(m,j)y^j+\dots+(m,m)y^m = (1+y)(1+y^3)\dots(1+y^{2m-1}).$$

What corresponds to this identity for $y = \pm 1$?

None

Question (1955 STEP II Q301)

Show that

$$\frac{2^{2n}}{2n} < \frac{(2n)!}{(n!)^2} < 2^{2n}$$

and, by induction or otherwise, that

$$\frac{(2n)!}{(n!)^2} < 2^{2n-1}$$

for $n > 5$. Deduce an inequality for

$$\prod_{\substack{m<p<2m \\ p \text{ prime}}} p$$

and hence, or otherwise, show that

$$\prod_{\substack{p \leq m \\ p \text{ prime}}} p < 2^{2m}$$

for all positive integers m.

None

Question (1948 STEP IIII Q303)

Prove that, for any positive integer n,

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + x^n,$$

where

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

Deduce the identity

$$1 + \sum_{r=1}^{m-1} (-1)^r \binom{2m-1}{r} = \frac{(-1)^{m+1}}{2}\left\{\binom{2m}{m} - \binom{2m}{m-1}\right\},$$

where m is any positive integer.

Question (1947 STEP II Q404)

A number, n, of different objects are divided into two groups containing r and $n-r$ members. If the objects of the two separate groups so formed are then permuted amongst themselves and give $N(n,r)$ different permutations of the n objects, prove that

$$\sum_{r=0}^{n} \{N(n,r)\}^{-1} = \frac{2^n}{n!}.$$

Question (1922 STEP I Q104)

If

$$(1+x)^n = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

prove that

$$a_0^2 + a_1^2 + a_2^2 + \cdots + a_n^2 = \frac{(2n)!}{n!n!},$$

and find the value of

$$a_0^2 - a_1^2 + a_2^2 - \cdots + (-1)^n a_n^2.$$

Question (1926 STEP I Q104)

Find numerical values of a, b, c such that the expansions of

$$(1+x)^n + b\left(1+\frac{x}{4}\right)^{4n} \text{ and } a\left(1+\frac{x}{2}\right)^{2n} + c\left(1+\frac{x}{8}\right)^{8n}$$

may be identical for the first four terms.

$$[1]: \quad 1 + b = a + c$$

$$[x]: \quad 1 + b = a + c$$

$$[x^2]: \quad \frac{n(n-1)}{2} + \frac{4n(4n-1)}{2}\frac{b}{16} = \frac{2n(2n-1)}{2}\frac{a}{4} + \frac{8n(8n-1)}{2}\frac{c}{8^2}$$

$$(n-1) + \left(n - \frac{1}{4}\right) b = \left(n - \frac{1}{2}\right) a + \left(n - \frac{1}{8}\right) c$$
$$1 + \frac{1}{4} b = \frac{1}{2} a + \frac{1}{8} c$$
$$[x^3]: \quad \frac{n(n-1)(n-2)}{6} + \frac{4n(4n-1)(4n-2)}{6}\frac{b}{4^3} = \frac{2n(2n-1)(2n-2)}{2}\frac{a}{2^3} + \frac{8n(8n-1)(8n-2)}{2}\frac{c}{8^3}$$
$$(n-1)(n-2) + \left(n - \frac{1}{4}\right)\left(n - \frac{1}{2}\right) b = \left(n - \frac{1}{2}\right)(n-1)\, a + \left(n - \frac{1}{8}\right)\left(n - \frac{1}{4}\right) c$$
$$2 + \frac{1}{8} b = \frac{1}{2} a + \frac{1}{32} c$$
$$\Rightarrow \quad (a, b, c) = (7, 14, 8)$$

Question (1927 STEP I Q104)

If
$$(1+x)^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots,$$
where n is a positive integer, shew by considering the product of $(1+x)^n$ and $\left(1+\frac{1}{x}\right)^n$, or otherwise, that the sum of
$$a_0 a_1 + a_1 a_2 + a_2 a_3 + \ldots\ldots$$
is
$$\frac{(2n)!}{(n+1)!(n-1)!},$$
and find the value of
$$a_0 a_1 - a_1 a_2 + a_2 a_3 - \ldots\ldots$$
where n is the odd integer $2p+1$.

Question (1928 STEP I Q105)

Prove the Binomial Theorem for a positive integral index. If the binomial expansion of $(1+x)^m$, where m is a positive integer, be written
$$(1+x)^m = 1 + p_1 x + p_2 x^2 + \cdots + p_m x^m,$$
shew that
$$1 + p_1 + p_2 + \cdots + p_m = 2^m,$$
$$p_1 + 2p_2 + 3p_3 + \cdots + m p_m = m 2^{m-1},$$
and find the value of
$$1 + p_1^2 + p_2^2 + \cdots + p_m^2.$$

Question (1932 STEP I Q102)

Shew that, if c_r is the coefficient of x^r in the expansion of $(1+x)^n$, where n is a positive integer,

$$c_0^2 + c_1^2 + c_2^2 + \cdots + c_n^2 = \frac{(2n)!}{(n!)^2}.$$

Find the value of $c_0^2 - c_1^2 + c_2^2 - \cdots + c_n^2$, where n is even.

Question (1938 STEP I Q102)

Prove that, if $(1+x)^n = c_0 + c_1 x + \cdots + c_n x^n$, then

$$c_0c_2 + c_1c_3 + \cdots + c_{n-2}c_n = \frac{(2n)!}{(n-2)!(n+2)!},$$

and

$$\frac{c_0}{2} + \frac{c_1}{3} + \cdots + \frac{c_n}{n+2} = \frac{2^{n+1}n+1}{(n+1)(n+2)}.$$

Question (1919 STEP I Q103)

Show that, if x and y are positive, m and n positive integers, and if the greatest term of the Binomial expansion of $(x+y)^m$ is the pth, and the greatest term of that of $(x+y)^n$ is the qth, then the greatest term of the expansion of $(x+y)^{m+n}$ is the $(p+q)$th or the $(p+q-1)$th.

Question (1915 STEP I Q106)

Assuming the binomial theorem for a positive or negative integral exponent, show that the coefficient of x^n in the expansion of

$$(1 + x + x^2 + \cdots + x^{n-1})^p,$$

where p is a positive integer, is

$$\frac{(p+n-1)!}{(p-1)!n!}p.$$

Question (1919 STEP I Q102)

Prove that the sum of the odd coefficients in the binomial expansion is equal to the sum of the even coefficients, and each is equal to 2^{n-1}, where n (a positive integer) is the index of the expansion.

Question (1935 STEP I Q102)

Prove that

$$\sum_{r=0}^{n} {}^nC_r \left(r - \frac{1}{2}n\right)^2 = 2^{n-2}n,$$

where n is a positive integer and ${}^nC_r$ is the coefficient of x^r in the expansion of $(1+x)^n$. Hence, or otherwise, prove that, if $S(n)$ is the sum of all the coefficients ${}^nC_r$ and $S_\delta(n)$ the sum of those coefficients ${}^nC_r$ for which r/n lies outside the interval $(\frac{1}{2} - \delta, \frac{1}{2} + \delta)$, where δ is a number satisfying $0 < \delta < \frac{1}{2}$, then

$$\frac{S_\delta(n)}{S(n)} < \frac{1}{4\delta^2 n}.$$

Question (1914 STEP II Q203)

Prove that in the expansion of $(1+x)^m + (1-x)^m$, where $-1 < x < 1$, the terms are either all positive or after a stage become and remain negative, and that in the latter case the last positive term exceeds numerically the sum of the infinite series of negative terms. Deduce or otherwise prove that $(1+x)^m + (1-x)^m - 2$ vanishes with or has the same sign as $m(m-1)x^2$.

None

Question (1927 STEP II Q201)

If

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n,$$

prove that

(i) $c_0 - c_1 + c_2 - c_3 + \cdots + (-1)^r c_r = \frac{(-1)^r(n-1)!}{r!(n-r-1)!} \quad (r < n)$,

(ii) $c_0c_1 + c_1c_2 + \cdots + c_{n-1}c_n = \frac{2n!}{(n-1)!(n+1)!}$.

If

$$(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n},$$

prove that

$$a_0 + a_3 + a_6 + \cdots = a_1 + a_4 + a_7 + \cdots = a_2 + a_5 + a_8 + \ldots.$$

Question (1932 STEP II Q203)

Prove that, if $(1+x)^n = c_0 + c_1x + \cdots + c_nx^n$, then

$$c_0c_2 + c_1c_3 + \cdots + c_{n-2}c_n = \frac{(2n)!}{(n-2)!(n+2)!},$$

$$\frac{c_0}{1} - \frac{c_1}{2} + \frac{c_2}{3} - \cdots + (-1)^n \frac{c_n}{n+1} = \frac{1}{n+1},$$

$$\frac{c_0}{1^2} - \frac{c_1}{2^2} + \frac{c_2}{3^2} - \cdots + (-1)^n \frac{c_n}{(n+1)^2} = \frac{1}{n+1}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right).$$

Question (1935 STEP II Q203)

Prove that, if $(1+x)^n = c_0 + c_1 x + \cdots + c_n x^n$, then

1. $c_0^2 - c_1^2 + c_2^2 - \cdots \pm c_n^2$ is equal to $(-1)^m (2m)!/(m!)^2$ if n is an even integer $2m$, and is zero if n is an odd integer.
2. $\frac{c_0}{y} - \frac{c_1}{y+1} + \frac{c_2}{y+2} - \cdots \pm \frac{c_n}{y+n} = \frac{n!}{y(y+1)\ldots(y+n)}$, where y is not zero or a negative integer.

Question (1937 STEP I Q303)

If

$$u = (a-b)^n + (b-c)^n + (c-a)^n,$$

where n is a positive integer, prove that

(i) If u is divisible by $\Sigma a^2 - \Sigma bc$ then n is of the form $3k \pm 1$, where k is an integer.

(ii) If u is divisible by $(\Sigma a^2 - \Sigma bc)^2$ then n is of the form $3k+1$.

Question (1935 STEP III Q304)

If ${}^nC_r$ denotes the number of combinations of n things taken r at a time, establish the following results by considering $(1-x^2)^{2n}$, or otherwise:

$$\frac{1}{2}{}^{2n}C_r({}^{2n}C_r - 1) = {}^{2n}C_{r-1}{}^{2n}C_{r+1} - {}^{2n}C_{r-2}{}^{2n}C_{r+2} + \cdots + (-1)^{r-1}{}^{2n}C_{2r},$$

$$ {}^{2n}C_r = {}^nC_r + 2^{r-2n}C_{r-2}{}^{n-r+2}C_1 + 2^{r-4n}C_{r-4}{}^{n-r+4}C_2 + \ldots,$$

the last term being ${}^nC_{\frac{r}{2}}$ or $2^{n-1}C_{\frac{r-1}{2}}$ according as to whether r is even or odd.

Question (1926 STEP I Q404)

(i) If c_r is the coefficient of x^r in the expansion of $(1+x)^n$ in a series of ascending powers of x, prove that

$$c_1^2 + 2c_2^2 + 3c_3^2 + \cdots + nc_n^2 = \frac{2n-1!}{n-1!n-1!}.$$

(ii) If a and b are unequal, and if $\frac{a-bx}{(1-x)^2}$ is equal to the sum of the first r terms of its expansion in a series of ascending powers of x, prove that

$$x = \frac{a + r(a-b)}{b + r(a-b)}.$$

Question (1921 STEP II Q403)

If

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \dots$$

find

$$c_0 - c_1 + c_2 - \dots + (-1)^n c_n.$$

Prove that the coefficient of x^n in the expansion in powers of x of

$$\frac{1}{(1-x)(1-x^2)(1-x^5)}$$

is $(n+1)^2$. (Note: The original text had a typo in the denominator, x^5 vs x^3. Transcribed as written. Also, the claim regarding the coefficient seems incorrect for the given function.)

Question (1924 STEP II Q403)

If $(1+x+x^2)^n = 1 + c_1 x + c_2 x^2 + \dots + c_{2n} x^{2n}$, where n is a positive integer, prove that

1. $c_n = 1 - c_1^2 + c_2^2 - c_3^2 + \dots + c_{2n}^2$,
2. $c_r = {}_nC_r + \frac{r-1}{1!}{}_nC_{r-1} + \frac{(r-2)(r-3)}{2!}{}_nC_{r-2} + \dots$,

and find the last term, where ${}_nC_s$ is the coefficient of x^s in $(1+x)^n$.

Question (1919 STEP III Q404)

If ${}_nC_r$ is the coefficient of x^r in the expansion of $(1+x)^n$ by the binomial theorem where n is a positive integer, prove that

$$\sum_{s=0}^{s=n-r} {}_rC_s \cdot {}_{n-r}C_s = {}_{2n-r}C_n.$$

Question (1937 STEP III Q402)

Using $\binom{x}{r}$ to denote $\frac{x(x-1)(x-2)\dots(x-r+1)}{1.2.3\dots r}$ for positive integral values of r, prove that

$$\binom{x+y}{r} = \binom{x}{r} + \binom{x}{r-1}\binom{y}{1} + \binom{x}{r-2}\binom{y}{2} + \dots + \binom{y}{r},$$

$$\binom{x+r+1}{r} = \binom{x+r-1}{r} + 2\binom{x+r-2}{r-1} + 3\binom{x+r-3}{r-2} + \dots + r\binom{x}{1} + r + 1.$$

Question (1938 STEP III Q402)

Shew that if n be a positive integer:

1. $n - \frac{n^2(n-1)}{1!2!} + \frac{n^2(n^2-1^2)(n-2)}{2!3!} - \cdots + (-1)^{n-1}\frac{n^2(n^2-1^2)\ldots(n^2-(n-2)^2)\cdot 1}{(n-1)!n!} = 0.$
2. $n - \frac{n(n^2-1^2)}{1!2!} + \frac{n(n^2-1^2)(n^2-2^2)}{2!3!} - \cdots + (-1)^{n-1}\frac{n(n^2-1^2)\ldots(n^2-(n-1)^2)}{(n-1)!n!} = (-1)^{n-1}.$

Question (1914 STEP IIII Q402)

If n be an integer and P_n the product of all the coefficients in the expansion of $(1+x)^n$, prove that $P_{n+1}/P_n = (n+1)^n/n!$. Prove that the coefficient of x^n in the expansion of

$$(1+x^{2^n})/(1-x) \text{ is } 2^{n-1}(n^2+4n+2).$$

This seems to be a typo in the original paper, as the question is non-sensical. A likely intended question is "Prove that the coefficient of x^n in the expansion of $(1+x)^{2n}/(1-x)$ is ...". The OCR seems to match the original, however.

Question (1920 STEP II Q502)

Prove the Binomial Theorem for a positive integral exponent. If c_r is the coefficient of x^r in the expansion of $(1-x)^n$, prove that

$$1 + c_1 + c_2 + \cdots + c_r = \frac{n+1}{n}\left(1 - \frac{1}{c_r - c_{r+1}}\right),$$

and that

$$1 + \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_r} = \frac{n+1}{n+2}\left\{1 - \frac{1}{c_{r+1} - c_r}\right\}.$$

Question (1924 STEP II Q505)

Prove that the sum of the first $r+1$ coefficients in the expansion of $(1-x)^{-n}$ by the binomial theorem, n being a positive integer, is

$$\frac{(n+r)!}{n!r!}.$$

Prove that the number of ways in which n prizes may be distributed among q people so that everybody may have one *at least* is

$$q^n - q(q-1)^n + \frac{q(q-1)}{2!}(q-2)^n - \ldots.$$

Question (1933 STEP III Q502)

If $\binom{n}{r}$ denotes the number of combinations of n things taken r at a time, shew that

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1} \text{ and derive an interpretation for } \binom{n}{0}.$$

Prove that

$$\binom{m+n}{r} = \binom{m}{r}\binom{n}{0}+\binom{m}{r-1}\binom{n}{1}+\binom{m}{r-2}\binom{n}{2}+\cdots+\binom{m}{1}\binom{n}{r-1}+\binom{m}{0}\binom{n}{r}.$$

By means of the identity $1-x^6=(1-x^3)(1+x+x^2)$, or otherwise, prove that

$$\binom{6m}{n}\binom{6m}{n}+\binom{6m-1}{n}\binom{6m-2}{n}+\binom{6m-2}{n}\binom{6m-4}{n}+\cdots+\binom{3m}{n}\binom{0}{n}$$

$$=\binom{n+6m-1}{6m-1}\binom{n}{0}-\binom{n+6m-4}{6m-3}\binom{n}{1}+\binom{n+6m-7}{6m-6}\binom{n}{2}-\cdots+(-1)^m\binom{n+1}{0}\binom{n}{2m}.$$

Question (1915 STEP IIII Q503)

If

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n,$$

where n is a positive integer, find $c_0^2 + c_1^2 + \cdots + c_n^2$.

If

$$s_0 = c_0 + c_3 + c_6 + \ldots,$$
$$s_1 = c_1 + c_4 + c_7 + \ldots,$$
$$s_2 = c_2 + c_5 + c_8 + \ldots,$$

prove that

$$s_0^2 + s_1^2 + s_2^2 = 1 + s_1 s_2 + s_2 s_0 + s_0 s_1.$$

Question (1914 STEP II Q602)

State and prove the Binomial Theorem for a positive integral exponent. If

$$(1+x)^{4m} = 1 + c_1 x + c_2 x^2 + \ldots,$$

where m is a positive integer, prove that

$$1 + c_1 + c_2 + c_3 + \cdots = 2^{4m}$$

and that

$$1 - c_2 + c_4 - c_6 + \cdots = (-1)^m \cdot 2^{2m}.$$

Question (1924 STEP II Q602)

If $(1+x)^n = c_0 + c_1x + \cdots + c_nx^n$, where n is a positive integer, prove that

1. $c_0^2 + c_1^2 + \cdots + c_n^2 = (2n)!/(n!)^2$,
2. $c_1 + 2c_2 + \cdots + nc_n = n2^{n-1}$,
3. $\frac{c_0}{1^2} - \frac{c_1}{2^2} + \frac{c_2}{3^2} - \cdots + \frac{(-1)^n c_n}{(n+1)^2} = \frac{1}{n+1}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right)$.

Question (1921 STEP III Q604)

Prove that, if $(1+x)^n = p_0 + p_1x + \cdots + p_nx^n$, where n is a positive integer,

$$\frac{p_0}{n+1} - \frac{1}{2}\frac{p_1}{n+2} + \cdots + (-1)^r \frac{p_r}{n+r+1} + \cdots = \frac{2^n(n!)^2}{(2n+1)!}.$$

Question (1922 STEP I Q706)

If q_r denote the number of combinations of n things r at a time, prove from first principles that the number of combinations of $(n+1)$ things r at a time is $q_r + q_{r-1}$. Prove also that $q_1 - \frac{1}{2}q_2 + \frac{1}{3}q_3 - \cdots + (-1)^{n-1}\frac{1}{n}q_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Question (1923 STEP I Q708)

If

$$(1 + px + x^2)^n = 1 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n},$$

prove that

$$a_r = a_{2n-r}.$$

Shew that

$$1 + 3a_1 + 5a_2 + \cdots + (4n+1)a_{2n} = (2n+1)(2+p)^n.$$

Question (1919 STEP II Q704)

If $(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots$ when n is a positive integer, find

(i) $c_0^2 + c_1^2 + \cdots + c_n^2$,

(ii) $\frac{c_0}{2} + \frac{c_1}{3} + \cdots + \frac{c_n}{n+2}$.

Question (1923 STEP III Q704)

If $(1+x)^n = c_0 + c_1x + \cdots + c_nx^n$, prove that

$$\frac{c_0}{n+1} - \frac{c_1}{n+2} + \frac{c_2}{n+3} - \cdots + (-1)^n \frac{c_n}{2n+1} = \frac{(n!)^2}{(2n+1)!}.$$