Question (1971 STEP I Q1)

Show that, if the polynomial

$$f(x) = x^3 + 3ax + b \quad (a \neq 0)$$

can be expressed in the form

$$A(x-p)^3 + B(x-q)^3,$$

where A and B are constants, then $p \neq q$, and p, q are the roots of the equation

$$at^2 + bt - a^2 = 0.$$

Prove, conversely, that if this equation has unequal roots then $f(x)$ can be written in the form (1). Hence or otherwise find the real root of the equation

$$x^3 + 54x + 54 = 0.$$

None

Question (1975 STEP I Q3)

The quartic equation $x^4 - s_1x^3 + s_2x^2 - s_3x + s_4 = 0$ has roots α, β, γ, δ. Find the cubic equation with roots $\alpha\beta + \gamma\delta$, $\alpha\gamma + \beta\delta$, $\alpha\delta + \beta\gamma$. Supposing that methods of solving quadratic and cubic equations are known, describe a procedure for solving a quartic equation.

Note that:

- $$\begin{aligned}\sum r &= \alpha\beta + \gamma\delta + \alpha\gamma + \beta\delta + \alpha\delta + \beta\gamma \\ &= s_2\end{aligned}$$

- $$\begin{aligned}\sum rs &= (\alpha\beta + \gamma\delta)(\alpha\gamma + \beta\delta) + (\alpha\gamma + \beta\delta)(\alpha\delta + \beta\gamma) + (\alpha\delta + \beta\gamma)(\alpha\beta + \gamma\delta) \\ &= \sum_{\text{sym}} \alpha^2\beta\gamma \\ &= \left(\sum \alpha\right)\left(\sum \alpha\beta\gamma\right) - 4\alpha\beta\gamma\delta \\ &= s_1s_3 - 4s_4\end{aligned}$$

- $$\begin{aligned}rst &= (\alpha\beta + \gamma\delta)(\alpha\gamma + \beta\delta)(\alpha\delta + \beta\gamma) \\ &= (\alpha^2\beta\gamma + \alpha\beta^2\delta + \alpha\gamma^2\delta + \beta\gamma\delta^2)(\alpha\delta + \beta\gamma) \\ &= \alpha^3\beta\gamma\delta + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha\beta\gamma\delta^3 + \alpha^2\beta^2\gamma^2 + \alpha\beta^3\delta\gamma + \alpha\beta\gamma^3\delta + \beta^2\gamma^2\delta^2 \\ &= \left(\sum \alpha^2\right)\alpha\beta\gamma\delta + \left(\sum \alpha\beta\gamma\right)^2 - 2\alpha\beta\gamma\delta\left(\sum \alpha\beta\right) \\ &= (s_1^2 - 2s_2)s_4 + s_3^2 - 2s_4s_2 \\ &= s_1^2s_4 + s_3^2 - 4s_2s_4\end{aligned}$$

Therefore we have a cubic $x^3 - s_2x^2 + (s_1s_3 - 4s_4)x - (s_1^2s_4 + s_3^2 - 4s_2s_4)$.
Notice that $(\alpha+\beta)(\gamma+\delta) = \alpha\gamma + \beta\delta + \alpha\delta + \beta\gamma$.
Therefore our strategy for solving our quartic will be as follows:

1. Form the corresponding cubic, and find the 3 roots.
2. Take a pair of two roots and we can find a quadratic with roots $\alpha+\beta$ and $\gamma+\delta$.
3. Using a different quadratic, we can find $\alpha\beta$ and $\gamma\delta$ (using the remaining root and the fact we know the full product).

Now we have $\alpha\beta$ and $\alpha+\beta$ we can solve to find all roots.

Question (1984 STEP I Q2)
The cubic equation
$$x^3 + ax^2 + bx + c = 0$$
has roots α, β, γ. Find a cubic with roots $\alpha^3, \beta^3, \gamma^3$, its coefficients being expressed in terms of a, b and c.

None

Question (1965 STEP I Q4)
The equation
$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$
where a and e are not zero, has roots $\alpha, \beta, \gamma, \delta$. Show how it is possible to obtain $\alpha^n + \beta^n + \gamma^n + \delta^n$ in terms of the coefficients a, b, c, d, e for all values of n, where n is a positive or negative integer. Obtain the equations whose roots are

(i) $\alpha^2, \beta^2, \gamma^2, \delta^2$;

(ii) $\alpha - 3, \beta - 3, \gamma - 3, \delta - 3$.

None

Question (1958 STEP I Q103)
If a, b and c are the roots of the equation
$$x^3 + px^2 + qx + r = 0,$$
form the equation with the roots $a^3 - bc$, $b^3 - ca$, and $c^3 - ab$. Prove that one of the roots is the geometric mean of the other two if, and only if, $rp^3 = q^3$. Find in a similar way a condition for one root to be the arithmetic mean of the other two. What can be said about a, b and c if both these conditions hold?

None

Question (1964 STEP IIII Q208)

Given that the roots of the equation

$$y^8 + 3y^2 + 2y - 1 = 0$$

are the fourth powers of the roots of an equation

$$x^8 + ax^2 + bx + c = 0$$

with rational coefficients a, b, c, find suitable values for a, b, c.

None

Question (1958 STEP II Q402)

(i) Find the equation whose roots are the cubes of the roots of the equation

$$x^3 + ax^2 + bx + c = 0.$$

(ii) Show how to obtain the equation whose roots are the roots of a given algebraic equation multiplied by the same constant quantity. Hence, or otherwise, prove that an algebraic equation with integer coefficients and unit coefficient for the highest power cannot have a real rational root which is not integral.

None

Question (1957 STEP IIII Q104)

Find the equation whose roots are the squares of the roots of the cubic equation $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$. If the values of the coefficients a_0, a_1, a_2, a_3 are given numerically, and the roots α, β, γ of the equation are real, and such that $|\alpha| > |\beta| > |\gamma|$, show that the continued repetition of this process will yield an approximate value of $|\alpha|$. Suggest a method of finding the other roots.

Question (1955 STEP IIII Q201)

Show that the result of eliminating y and z between the three equations

$$y^2 + 2ay + b = 0, \quad (1)$$
$$z^2 + 2cz + d = 0, \quad (2)$$
$$x = yz,$$

is the quartic equation whose roots are $\alpha\beta, \alpha\beta', \alpha'\beta, \alpha'\beta'$, where α, α' are the roots of the quadratic equation (1) and β, β' are the roots of the quadratic equation (2). Find the quartic equation whose roots are

$$(\alpha - \beta), (\alpha' - \beta), (\alpha - \beta'), (\alpha' - \beta'),$$

and hence write down the equation whose roots are

$$(\alpha - \beta)^2, (\alpha' - \beta)^2, (\alpha - \beta')^2, (\alpha' - \beta')^2.$$

None

Question (1956 STEP IIII Q204)

Find the equation whose roots are less by 2 than the squares of the roots of

$$x^3 + qx + r = 0.$$

Examine the particular case

$$x^3 - 3x + 1 = 0,$$

and interpret the result.

Question (1955 STEP IIII Q301)

The equation

$$x^3 + px^2 + qx + r = 0$$

has roots α, β, γ. Find the equations with roots (i) $\beta+\gamma, \gamma+\alpha, \alpha+\beta$, (ii) $\beta\gamma, \gamma\alpha, \alpha\beta$. Hence or otherwise determine necessary and sufficient conditions for the equation

$$x^3 + px^2 + qx + r = 0$$

to have two roots (i) whose sum is a, (ii) whose product is b.

Question (1956 STEP IIII Q301) (i) If a_1, a_2, a_3 are the roots of

$$x^3 + px + q = 0,$$

where $p + q + 1 \neq 0$, find the equation with roots

$$1/(1 - a_i) \quad (i = 1, 2, 3).$$

(ii) Solve the equations

$$\begin{aligned} x + y + z &= 7, \\ x^2 + y^2 + z^2 - 4z &= 5, \\ xyz - 2xy &= 4. \end{aligned}$$

Question (1957 STEP IIII Q302)

Let $z_1, \dots, z_n$ be the zeros of

$$f(z) = z^n + c_1 z^{n-1} + \dots + c_{n-1} z + c_n$$

and let $f'(z)$ be the derivative. Show that

$$(-1)^{\frac{1}{2}n(n-1)} \prod_{1 \le i < j \le n} (z_i - z_j)^2 = \prod_{i=1}^{n} f'(z_i).$$

Hence, or otherwise, show that if $y_1, \dots, y_n$ are the zeros of

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

then

$$\prod_{1 \le i < j \le n} (y_i - y_j)^2 = (-1)^{\frac{1}{2}n(n-1)} (n!)^n.$$

None

Question (1952 STEP II Q402)

The cubic equation $x^3 + px + q = 0$ has roots α, β, γ. Find the cubic equation whose roots are $(\beta - \gamma)^2, (\gamma - \alpha)^2, (\alpha - \beta)^2$. Hence or otherwise deduce the condition for the cubic equation

$$a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0$$

to have a pair of equal roots in the form

$$(a_0^2 a_3 + 2a_1^3 - 3a_0 a_1 a_2)^2 + 4(a_0 a_2 - a_1^2)^3 = 0.$$

Question (1955 STEP II Q401)

Find the equation whose roots are the squares of the roots of the cubic equation

$$x^3 - ax^2 + bx - 1 = 0.$$

Find all pairs of values of a and b for which the two equations are the same.

Question (1946 STEP IIII Q303)

If $a_r = x + (r-1)y$, show that

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{vmatrix} = (-ny)^{n-1} \{x + \frac{1}{2}(n-1)y\}.$$

Question (1945 STEP II Q403)

Form the equation whose roots are the reciprocals of the roots of the equation

$$x^3 + ax^2 + bx - c = 0.$$

Hence solve the equation

$$35x^3 - 18x^2 + 1 = 0.$$

Question (1944 STEP II Q204)

Prove that the geometric mean of n positive numbers is less than or equal to their arithmetic mean. Shew that, if the equation

$$x^n - nb_1x^{n-1} + \cdots + \frac{n(n-1)\dots(n-r+1)}{r!}(-b_r)^r x^{n-r} + \cdots + (-b_n)^n = 0,$$

where $b_1, b_2, \dots, b_n$ are all real and greater than zero, has n real roots, then $b_r \leq b_{r-1}$ for $r = 1, 2, \dots, n-1$.

Question (1927 STEP I Q111)

Shew that

$$\frac{d^n}{dx^n}(\tan^{-1} x) = (-1)^{n-1}(n-1)!r^{-n}\sin n\phi,$$

where

$$r\cos\phi = x, \quad r\sin\phi = 1.$$

Question (1929 STEP I Q104)

Reduce the equation $x^3 + 3px^2 + 3qx + r = 0$ to the form $y^3 + 3y + m = 0$ by assuming $x = \lambda y + \mu$; and solve this equation by assuming $y = z - \frac{1}{z}$. Hence prove the condition for equal roots to be

$$4(p^2 - q)^3 = (2p^3 - 3pq + r)^2.$$

Question (1935 STEP II Q202)

If

$$\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}=1,$$
$$\frac{x}{a+\mu}+\frac{y}{b+\mu}+\frac{z}{c+\mu}=1,$$
$$\frac{x}{a+\nu}+\frac{y}{b+\nu}+\frac{z}{c+\nu}=1,$$

prove that, for all values of ξ (except $-a, -b$ and $-c$),

$$\frac{x}{a+\xi}+\frac{y}{b+\xi}+\frac{z}{c+\xi}=1+\frac{(\lambda-\xi)(\mu-\xi)(\nu-\xi)}{(a+\xi)(b+\xi)(c+\xi)},$$

and that

$$x=\frac{(a+\lambda)(a+\mu)(a+\nu)}{(a-b)(a-c)}.$$

Question (1937 STEP II Q203)

If $u_0 = 2$, $u_1 = 2\cos\theta$, and

$$u_n = u_1 u_{n-1} - u_{n-2}, \quad (n > 1)$$

prove that $u_n = 2\cos n\theta$. By successive applications of the relation express u_5 as a polynomial in u_1, say $f(u_1)$. Then by considering the equation $f(u_1) = 1$ shew that $2\cos\frac{\pi}{15}$ is a root of the quartic equation

$$x^4 + x^3 - 4x^2 - 4x + 1 = 0.$$

What are the other three roots?

Question (1940 STEP II Q208)

(i) If $u = xyz$, where x, y, z are connected by the relations

$$yz + zx + xy = a, \quad x + y + z = b \quad (a, b \text{ being constants}),$$

prove that

$$\frac{du}{dx} = (x-y)(x-z).$$

(ii) If ξ, η are functions of x, y such that $\xi = e^x\cos y, \eta = e^x \sin y$, and x, y are functions of r, θ such that $x = e^r\cos\theta, y = e^r\sin\theta$, where r is a function of θ, prove that

$$\frac{d\xi}{d\eta} = \frac{\frac{dr}{d\theta} - \tan(y+\theta)}{1 + \frac{dr}{d\theta}\tan(y+\theta)}.$$

Question (1923 STEP IIII Q201)

Two pairs of points A, B and A', B' lie on an axis Ox, and their abscissae are given by the equations $ax^2 + 2bx + c = 0$ and $a'x^2 + 2b'x + c' = 0$ respectively. Find an equation with rational coefficients which has $AA' \cdot BB'$ for one of its roots. Give the geometrical interpretation of the relations obtained by equating the various coefficients in the equation to zero.

Question (1920 STEP III Q308)

Prove that

$$\frac{1}{0!2n!} - \frac{1}{1!3!2n-1!} + \frac{1}{2!4!2n-2!} - \cdots + (-)^{n+1}\frac{1}{n-1!n+1!} = \frac{1}{n-1!n+1!2n!}.$$

Question (1921 STEP III Q311)

A regular polygon of $2n+1$ sides is inscribed in a circle of radius a. From one corner perpendiculars are drawn to the sides; prove that their sum is

$$(2n+1)a\cos\frac{\pi}{2n+1}.$$

Question (1935 STEP III Q301)

If n is any positive integer, shew that n consecutive odd integers can be found not one of which is prime. Shew also that one such sequence has sum

$$\frac{1}{2}\left\{2n+2-\sqrt{2n+1}+n(n+2)\right\}.$$

Question (1936 STEP III Q302)

(i) Denoting the roots of the equation $x^4 - x + 1 = 0$ by x_1, x_2, x_3, x_4, shew that, if $y_r = x_r^3 + x_r$, $[r = 1, 2, 3, 4]$, then y_r satisfies the equation $y^4 - 3y^3 + 7y^2 - 7y + 5 = 0$.
(ii) Given that the sum of two of its roots is zero, solve completely the equation

$$4x^4 + 8x^3 + 13x^2 + 2x + 3 = 0.$$

Question (1917 STEP II Q404)

Prove that the number $(r+1)^2(r-1)$, when expressed in the scale of r, is multiplied by $r-1$ when its digits are reversed.

Question (1939 STEP III Q404)

If the equation $x^5 + 5a_4x^4 + 10a_3x^3 + 10a_2x^2 + 5a_1x + a_0 = 0$ has three equal roots each equal to the arithmetic mean of the other roots, prove that $a_0 = 10a_3a_4^3 - 9a_4^5$ and obtain similar expressions for a_1 and a_2 in terms of a_3 and a_4.

State and prove the converse theorem.

Question (1924 STEP II Q606)

Find the n real quadratic factors of $x^{2n} - 2a^n x^n \cos n\phi + a^{2n}$. Show that $\prod_{r=0}^{r=n-1} \left\{\cos\phi - \cos\frac{2r\pi}{n}\right\} + \prod_{r=0}^{r=n-1} \left\{1 - \cos\left(\phi + \frac{2r\pi}{n}\right)\right\} = 0$.