Question (1978 STEP II Q16)

Let $f(x)$ be a polynomial in x. Explain why, if z is an approximation to a root of $f(x)$, then $z - f(z)/f'(z)$ is often a closer approximation. By considering polynomials of the form $x^r + a$, and suitable real values of z_0, show that the iteration

$$z_n = z_{n-1} - f(z_{n-1})/f'(z_{n-1}) \quad (n = 1, 2, \dots)$$

may exhibit any of the following three behaviours.

1. For every real value of z_0 it approaches a root of $f(x)$.
2. For no real value of z_0 does it approach a root of $f(x)$.
3. For some, but not for all, real values of z_0 it approaches a root of $f(x)$.

Question (1959 STEP II Q405)

Explain Newton's method for approximation to the real roots of an equation, namely, that in *certain circumstances* if a is a first approximation to a root of the equation $f(x) = 0$, then

$$a - \frac{f(a)}{f'(a)}$$

is a better one. Apply this to the equation $\sin x = \lambda x$, where λ is a small positive quantity, and show that $\pi[1 - \lambda + \lambda^2 - \lambda^3(1 + \frac{1}{6}\pi^2)]$ is a better approximation to the root near π than π itself.

Question (1957 STEP II Q102)

The equation $f(x) = 0$, where $f(x)$ is a polynomial, has a root ξ such that $f'(\xi) \neq 0$. Show that, if ξ_1 is a sufficiently good approximation to ξ, then

$$\xi_2 = \xi_1 - \frac{f(\xi_1)}{f'(\xi_1)}$$

is a better approximation to ξ. Use the above formula to evaluate $\sqrt[3]{3}$ to three decimal places.

Question (1951 STEP II Q405)

Justify Newton's method for approximating to a root of the equation $f(x) = 0$, namely, that if a is a first approximation, $a_1 = a - \frac{f(a)}{f'(a)}$ is in general a better approximation. Illustrate as simply as you can, graphically or otherwise, the general circumstances in which (i) a_1 is nearer to the actual root than a, (ii) the actual root lies between a and a_1. Consider the positive root between 1 and 2 of the equation $3\sin x = 2x$ by taking $a = \frac{\pi}{2}$. Find the next approximation and state which of the two cases mentioned above it illustrates.

Question (1945 STEP II Q409)

Justify Newton's method of approximation to the roots of the equation $f(x) = 0$ in the form $\alpha - f(\alpha)/f'(\alpha)$, where α is the first approximation, explaining by a diagram the importance of the equality of the signs of $f(\alpha)$ and $f''(\alpha)$, both assumed to be non-zero. Through a point on the circumference of a circle two chords are drawn to divide the area of the circle into three equal parts. Prove that the angle between the chords is approximately $30°44'$.

Question (1927 STEP IIII Q203)

If α is a first approximation to a root of an equation $f(x) = 0$, shew that $\alpha - \dfrac{f(\alpha)}{f'(\alpha)}$ is likely to be a better approximation. Discuss the conditions with reference to $f'(x), f''(x)$ which tend to vitiate this result. Explain the geometrical significance of your arguments. Apply the method to determine, correct to three places of decimals, the root of

$$x^3 - 8x - 60 = 0$$

which is nearly equal to 3.

Question (1934 STEP IIII Q205)

"If ξ is an approximate root of the equation $f(x) = 0$, then in general $\xi - f(\xi)/f'(\xi)$ is a better approximation."

Discuss this statement graphically, pointing out cases when repeated application of the approximation will give the value of the root to any desired degree of accuracy and cases when it will not.

If a is small the equation $\sin x = ax$ has a root ξ, nearly equal to π. Shew that

$$\xi = \pi \left\{1 - a + a^2 - \left(\frac{\pi^2}{6} + 1\right) a^3\right\}$$

is a better approximation, if a is sufficiently small.

Question (1935 STEP III Q305)

Establish Newton's method of approximating to the roots of an equation. Shew that between any two consecutive even integers there is one and only one real root of the equation $\frac{1}{2x} = \tan\frac{\pi x}{2}$. Prove that for a large value of n, the root between $2n$ and $2(n+1)$ is approximately $2n + \frac{1}{2\pi n}$. Prove similar results for the equation $4x = \tan\frac{\pi x}{2}$, with the result $2n + 1 - \frac{1}{2\pi(2n+1)}$.