Question (1974 STEP II Q10)

A simple pendulum has length l and is deflected through an angle $\theta(t)$ from the vertical. Without making any approximations, write down the equation of motion and deduce the equation of energy if α is the greatest value of θ reached. Show that the period is given by

$$2\left(\frac{l}{g}\right)^{\frac{1}{2}}\int_0^\alpha (\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta)^{-\frac{1}{2}}\,d\theta.$$

By making the substitution $\sin\frac{1}{2}\theta = \sin\frac{1}{2}\alpha\sin\psi$ and expanding the integrand appropriately, show that, for small values of α, the period is approximately

$$2\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}\left(1+\frac{1}{16}\alpha^2\right).$$

None

Question (1969 STEP III Q7)

Establish the equation of motion of a simple pendulum of length l in terms of the angle θ that the pendulum makes with the upward vertical. Deduce the equation expressing the conservation of energy. Find θ as a function of t given that, at time $t=0$, $\theta=\pi$ and the kinetic energy is $2mgl$; and show that the time taken for the pendulum to reach a position within a small angle α of the upward vertical is approximately $\sqrt{(l/g)}\ln(4/\alpha)$.

$$\left[\int \text{cosec}\ x\,dx = \ln\tan\frac{1}{2}x.\right]$$

None

Question (1981 STEP III Q12)

Consider a simple pendulum of length l and angular displacement θ which is not assumed to be small. Show that

$$\frac{1}{2}l\left(\frac{d\theta}{dt}\right)^2 = g(\cos\theta-\cos\gamma) \quad (1)$$

where γ is the maximum value of θ. Show also that the period P is given by

$$P = 2\sqrt{\frac{l}{g}}\int_0^\gamma \left(\sin^2(\gamma/2)-\sin^2(\theta/2)\right)^{-\frac{1}{2}}d\theta \quad (2)$$

By using the substitution $\sin(\theta/2)=\sin(\gamma/2)\sin\alpha$, or otherwise, show that for small values of γ the period is approximately

$$2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\gamma^2}{16}\right) \quad (3)$$

None

Question (1976 STEP IIII Q15)

Derive the equation for a simple pendulum

$$\ddot{\theta} = -\omega^2 \sin\theta,$$

giving a value for ω^2 in terms of relevant physical quantities. Show that for small α there is an approximate solution

$$\theta_1(t) = \alpha \sin \omega t. \tag{1}$$

By expanding $\sin\theta$ for small θ, and using the approximation (1) in the cubic term, obtain the higher-order approximation

$$\theta_2(t) = \alpha \sin \omega t - \frac{\alpha^3}{16}\omega t \cos \omega t + \frac{\alpha^3}{192}\sin 3\omega t, \tag{2}$$

for suitable starting conditions at $t = 0$. For how long an interval t would you expect this approximation to be reasonable? For a sufficiently small number of oscillations after $t = 0$ show that

$$\theta_3(t) = \alpha \sin \Omega t + \frac{\alpha^3}{192}\sin 3\Omega t, \tag{3}$$

where

$$\Omega = \omega(1 - \alpha^2/16).$$

Of the above two approximations (2) and (3), which do you prefer, and why?

None

Question (1963 STEP III Q105)

A circular groove of radius a is marked out on a plane inclined at an angle α to the horizontal. A particle is projected along the groove, from its lowest point, with velocity V_0. For all values of V_0, find in terms of a definite integral the time that elapses before the particle is again at the lowest point of the groove. Show that, if V_0 is large, the time is approximately

$$\frac{2\pi a}{V_0}\left[1 + \frac{ga}{4V_0^2}\sin^2\alpha\right].$$

(Frictional forces are to be neglected and it may be assumed that the particle does not leave the groove.)

None

Question (1961 STEP III Q308)

The period of small oscillations of a compound pendulum is T. It is hanging from a pivot and suddenly set in motion with angular velocity ω_0. Show that it makes complete revolutions in a vertical plane if $\omega_0 T > 4\pi$.

None

Question (1964 STEP III Q308)

In the finite motion of a simple pendulum of length l under gravity g, the inclination to the vertical oscillates between $-\alpha$ and $+\alpha$. Show that the total period of oscillation is given by

$$4(l/g)^{1/2}\int_0^{\pi/2}(1-\sin^2\frac{1}{2}\alpha\sin^2\psi)^{-1/2}d\psi.$$

If α is sufficiently small for the integrand to be expanded in powers of $\sin\frac{1}{2}\alpha$, show that to order α^2 the period is

$$2\pi(l/g)^{1/2}(1+\frac{1}{16}\alpha^2),$$

and find also the next term, in α^4.

None

Question (1914 STEP I Q111)

Find an expression for the velocity at any point in the path of a particle moving with simple harmonic motion. After the particle is 3 inches from the middle point of the path, moving away from the middle point, 4 seconds elapse until it is again in that position, moving towards the middle point, whilst a further 10 seconds elapses until it again arrives at that position. Find the length of the path.

Question (1941 STEP I Q109)

Obtain the equation of motion of a simple pendulum of length l,

$$l\frac{d^2\theta}{dt^2}+g\sin\theta=0,$$

and deduce that

$$\tfrac{1}{2}l\left(\frac{d\theta}{dt}\right)^2-g\cos\theta=\text{constant}.$$

If initially $\theta=0$ and $l\left(\frac{d\theta}{dt}\right)^2=4g$, prove that $\sin\frac{1}{2}\theta=\tanh\sqrt{\frac{g}{l}}t$. Illustrate by a rough graph how θ varies with t.

Question (1932 STEP I Q205)

Shew that the time of swing of a simple pendulum is independent of the amplitude if the cube of the ratio of the amplitude to the length is neglected. A pendulum of length 32 feet is drawn aside a distance of 1 foot and the bob is then projected towards the position of equilibrium with a velocity of 1 foot per second. Find the point at which the bob will first come to rest and the time from the moment of projection to that point.

Question (1920 STEP IIII Q210)

A simple pendulum of length l makes oscillations of angular extent α on each side of the vertical: find the equation expressing $d\theta/dt$ in terms of θ, the inclination of the string to the vertical at time t. If $\sin\phi = \sin\frac{1}{2}\theta/\sin\frac{1}{2}\alpha$, shew that the period of a complete swing (to and fro) is equal to

$$4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{(1-\sin^2\frac{1}{2}\alpha\sin^2\phi)}}.$$

The pendulum of a clock is calculated to have a period of 1 second for very small oscillations; shew that if the pendulum is kept swinging through an angle of 8° (so that $\alpha = 4^\circ$) the clock will lose about 26 seconds a day.