Question (1966 STEP I Q8)

Three distinct complex numbers z_1, z_2, z_3 are represented in the complex plane by points A_1, A_2, A_3. Prove that a necessary and sufficient condition for the triangle $A_1A_2A_3$ to be equilateral is

$$z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2.$$

None

Question (1976 STEP I Q6)

Let $z = \cos\theta + i\sin\theta$ $(\theta \neq \pi)$ and $w = (z-1)(z+1)^{-1}$. Show that w is purely imaginary, and hence show that the angle in a semi-circle is a right angle.

None

Question (1967 STEP IIII Q6)

Two distinct complex numbers z_1 and z_2 are given, with $|z_1| < 1$, $|z_2| < 1$. Prove that there is a positive real number K, depending on z_1 and z_2, such that

$$|1-z| \leq K(1-|z|)$$

for all complex numbers z whose representative points in the complex plane lie within, or on a side of, the triangle determined by the points representing z_1, z_2 and 1. Determine the smallest possible value of K in the case $z_1 = \frac{1}{2}(1+i)$, $z_2 = \frac{1}{2}(1-i)$.

None

Question (1965 STEP I Q6)

Three complex numbers z_1, z_2, z_3 are represented in the complex plane by the vertices of a triangle $A_1A_2A_3$. What is the locus of points representing the complex numbers $z_1 + it(z_2 - z_3)$, where t is a *real* parameter? Prove that the orthocentre of the triangle $A_1A_2A_3$ represents the complex number z, where

$$z = \frac{\bar{z}_1(z_2-z_3)(z_2+z_3-z_1) + \bar{z}_2(z_3-z_1)(z_3+z_1-z_2) + \bar{z}_3(z_1-z_2)(z_1+z_2-z_3)}{\bar{z}_1(z_2-z_3) + \bar{z}_2(z_3-z_1) + \bar{z}_3(z_1-z_2)}$$

and the bar indicates complex conjugate.

None

Question (1959 STEP I Q106)

On the sides of a triangle $Z_1Z_2Z_3$ are constructed isosceles triangles $Z_2Z_3W_1$, $Z_3Z_1W_2$, $Z_1Z_2W_3$ lying outside the triangle $Z_1Z_2Z_3$. The angles at W_1, W_2, W_3 are all $\frac{2\pi}{13}$. By assuming complex numbers z_1, z_2, z_3 to Z_1, Z_2, Z_3 and calculating the numbers representing W_1, W_2, W_3, or otherwise, prove that $W_1W_2W_3$ is equilateral.

None

Question (1964 STEP I Q106)

Four complex numbers are denoted by z_1, z_2, z_3, z_4. Show that their representative points in the complex plane are concyclic if and only if the cross-ratio

$$\frac{(z_1 - z_3)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

is real. Use this result to show that if these points are concyclic so are the points $1/z_2$, $1/z_3$, $1/z_4$.

None

Question (1961 STEP I Q301)

X, Y, Z are the centres of squares described externally on the sides of a triangle. Prove that AX, YZ are perpendicular and of equal length.

None

Question (1963 STEP IIII Q202)

Show that the condition that the two triangles in the Argand plane formed by the two triples of complex numbers a_1, a_2, a_3 and b_1, b_2, b_3 should be similar in the same sense is that

$$\frac{a_1 - a_3}{a_1 - a_2} = \frac{b_1 - b_3}{b_1 - b_2}.$$

The three triangles BCA', CAB', ABC' are similar in the same sense (although they are not necessarily similar to ABC). Show that the triangles ABC, $A'B'C'$ have the same centroid.

None

Question (1958 STEP IIII Q307)

The points z_1, z_2, z_3 form a triangle in the Argand diagram. Prove that it is equilateral if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2.$$

None

Question (1963 STEP II Q102)

Explain briefly how complex numbers may be represented as points in a plane. How many squares are there that have the points $3 - i$, $1 + 4i$ as two of their corners? In each case find the remaining two corners. If z_1, z_2 and z_3 are the complex numbers representing the vertices of an equilateral triangle, prove that

$$z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2.$$

If this condition is satisfied, what can you deduce about the points represented by z_1, z_2 and z_3, and why?

None

Question (1961 STEP II Q403)

The complex numbers a, b, c are represented in the Argand diagram by the points A, B, C. Show that ABC is an equilateral triangle if and only if a, b, c are not all equal and

$$a^2 + b^2 + c^2 - bc - ca - ab = 0.$$

Three equilateral triangles XYZ, YZX, ZXY are drawn outwards from the sides of a triangle XYZ. Show that the triangles XYZ, $X'Y'Z'$ have a common centre of gravity.

None

Question (1950 STEP I Q302)

Equilateral triangles BCD, CAE, ABF are constructed on the sides of a triangle ABC and external to this triangle. Prove that (i) the lines AD, BE, CF are concurrent; (ii) the circumcentres of the three given equilateral triangles are the vertices of another equilateral triangle.

Question (1954 STEP II Q102)

(i) a, b, c and d are distinct complex numbers. By an appeal to the Argand diagram or otherwise, show that, if any two of the numbers

$$\frac{a-b}{c-d}, \quad \frac{b-c}{a-d}, \quad \frac{c-a}{b-d}$$

are pure imaginaries, then so is the third. (ii) What complex numbers correspond, in the Argand diagram, to the centroid and the orthocentre of the triangle whose vertices are represented by the numbers $0, 1+3i$ and $5i$?

Question (1957 STEP II Q103)

Prove that the three (distinct) complex numbers z_1, z_2, z_3 represent the vertices of an equilateral triangle in the Argand diagram if and only if

$$z_1^2 + z_2^2 + z_3^2 - z_2z_3 - z_3z_1 - z_1z_2 = 0.$$

Question (1953 STEP II Q302)

The three complex numbers z_1, z_2, z_3 are represented in the Argand diagram by the vertices of a triangle $Z_1Z_2Z_3$ taken in counterclockwise order. On the sides of $Z_1Z_2Z_3$ are constructed isosceles triangles $Z_2Z_3W_1, Z_3Z_1W_2, Z_1Z_2W_3$, lying outside $Z_1Z_2Z_3$. The angles at W_1, W_2, W_3 all equal $2\pi/3$. Find the complex numbers represented by W_1, W_2, W_3 and prove that the triangle $W_1W_2W_3$ is equilateral.

Question (1944 STEP IIII Q102)

In the Argand diagram a triangle ABC is inscribed in the circle $|z| = 1$, the vertices A, B, C corresponding to the complex numbers a, b, c respectively. Prove that the orthocentre H is given by $z = a + b + c$. Verify that the circle

$$|2z - a - b - c| = 1$$

passes through the nine points from which it takes its name.

Question (1947 STEP II Q302)

Complex numbers $z_r (z_r = x_r + iy_r)$ are represented in the Argand diagram by points P_r with co-ordinates (x_r, y_r). Prove that

(i) a necessary and sufficient condition for the points P_1, P_2, P_3, P_4 to be concyclic is that

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

should be real;

(ii) a necessary and sufficient condition for the triangles $P_1P_2P_3$ and $P_4P_5P_6$ to be similar and with the same orientation is

$$\begin{vmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Deduce from (ii) a necessary and sufficient condition for the triangle $P_1P_2P_3$ to be equilateral.

None

Question (1916 STEP I Q102)

If a polygon of an even number of sides be inscribed in a circle, shew that the products of the perpendiculars drawn from any point on the circle on the alternate sides are equal.

Question (1913 STEP II Q207)

The roots of the quadratic equation $az^2 + 2bz + c = 0$, where a, b, c are real and $ac > b^2$, are represented on an Argand diagram by points P, Q. Prove that P and Q are equidistant from the origin, and that PQ is perpendicular to the axis of real numbers. Hence show that P and Q may be found by a geometrical construction which does not require the solution of the equation. Prove also that, if a', b', c' are real and $a'c' > b'^2$, the points representing the roots of $a'z^2 + 2b'z + c' = 0$ lie on the circle through P, Q and the origin, if $bc' = b'c$.