

Question (1968 STEP III Q5)

The integral

$$I = \int_{x-h}^{x+h} f(u) du$$

is to be approximated by an expression of the form $J = af(x-h) + bf(x) + cf(x+h)$, where a , b and c may depend on h but are independent of the function f and of x . Show that a , b and c may be chosen in such a way that $I = J$ whenever f is a polynomial of sufficiently low degree n , and find the largest n for which this is true. Find values of a , b , c such that $I = J$ whenever $f(u) = p + q \sin u + r \cos u$.

Question (1975 STEP III Q7)

Let $f_n(x) = (x^2 - 1)^n$ and let $\phi_n(x) = \frac{d^n}{dx^n} \{f_n(x)\}$. Use Leibniz' theorem on the differentiation of products to show that

$$\frac{d^r}{dx^r} \{f_n(x)\}$$

vanishes at $x = 1$ and $x = -1$ for all values of $r < n$. Hence show that $\int_{-1}^1 x^k \phi_n(x) dx = 0$ for all $k < n$, and deduce that if $m \neq n$ then

$$\int_{-1}^1 \phi_m(x) \phi_n(x) dx = 0.$$

Question (1979 STEP III Q3)

The real polynomial $f(x)$ has degree 5. Prove that

$$\int_{-y}^{+y} f(x) dx = \frac{1}{2} y \{f(\lambda y) + f(\mu y) + f(-\lambda y) + f(-\mu y)\}$$

for positive constants λ and μ (independent of y and f) whose squares are the roots of a certain quadratic equation to be determined.

Question (1963 STEP III Q210)

Show that there is a unique pair of real numbers a , b with the property that

$$\int_{-1}^{+1} P(x) dx = P(a) + P(b)$$

for all polynomials $P(x)$ of degree at most three.

Question (1962 STEP II Q109)

The function $L_n(x)$ is defined by

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}),$$

where n is a positive integer or zero. Show that $L_n(x)$ is a polynomial of degree n , that the coefficient of x^n is $(-1)^n$ and that $L_n(0) = n!$. By substituting for $L_n(x)$, but not for $L_m(x)$, and integrating by parts, or otherwise, show that

$$\int_0^\infty L_m(x)L_n(x)e^{-x}dx = \begin{cases} 0 & (n > m \geq 0), \\ (n!)^2 & (m = n). \end{cases}$$

Question (1963 STEP II Q301)

Write $f_n(x)$ for the polynomial $d^n/dx^n(x^2 - 1)^n$. Prove that if $k < n$

$$\int_{-1}^1 x^k f_n(x) dx = 0.$$

Deduce that, if $g_n(x) = d/dx\{(x^2 - 1)f'_n(x)\}$ and $k < n$,

$$\int_{-1}^1 x^k g_n(x) dx = 0.$$

Hence show that, if λ is the constant such that the coefficient of x^n in $h_n(x) = g_n(x) - \lambda f_n(x)$ vanishes, $h_n(x)$ is identically zero.

Question (1955 STEP II Q409)

$I(p, q)$ is defined as

$$\int_0^1 x^p(1-x)^q dx,$$

where p and q are real and non-negative. Show that

$$I(p, q) = I(q, p).$$

Obtain a reduction formula for the integral and state any limitations on the values of p and q necessary. Prove that if p and q are positive integers

$$I(p, q) = p!q!/(p+q+1)!$$

Question (1957 STEP II Q406)

The polynomial $f_n(x)$ is defined as $\frac{d^n}{dx^n}(x^2 - 1)^n$. Prove that all the roots of the equation $f_n(x) = 0$ are real and distinct and lie between ± 1 . Prove also that $\int_{-1}^1 f_n(x)f_m(x)dx = 0$ if $m \neq n$, and find its value when $m = n$.

Question (1944 STEP III Q307)

If

$$L_n(x) = e^x \frac{d^n}{dx^n} (x^n e^{-x}),$$

show that

$$(i) \quad xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0,$$

$$(ii) \quad \int_0^\infty e^{-x} x^k L_n(x) dx = 0$$

if k is an integer less than n .

Question (1924 STEP I Q112)

Prove by induction or otherwise that

$$\int_0^\pi \{\sin n\theta / \sin \theta\} d\theta = 0 \text{ or } \pi$$

according as n is an even or odd positive integer.

Question (1923 STEP I Q105)

Prove that, if

$$f_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} \{(x^2 - 1)^n\},$$

then

$$f_n(1) = 1, \quad f_n(-1) = (-1)^n.$$

Prove also, by integration by parts or otherwise, that, if $\phi(x)$ is a polynomial of degree less than n ,

$$\int_{-1}^1 \phi(x) f_n(x) dx = 0,$$

and that

$$\int_{-1}^1 f_n(x) f_m(x) dx = 0, \text{ or } 2/(2n+1),$$

according as $m \neq n$, or $m = n$.

Question (1920 STEP III Q204)

Give a discussion of the method of “proportional parts” as applied to interpolation in mathematical tables; and by considering the function

$$F(x) = f(x) - f(a) - \frac{x-a}{b-a}\{f(b) - f(a)\} - C(x-a)(x-b), \text{ or otherwise,}$$

shew that the error in the value of $f(c)$ as calculated from the tabular values given for $x = a, x = b$, is equal to

$$\frac{1}{2}(b-c)(c-a)f''(\gamma)$$

in excess of the true value, where c and γ lie between a and b . Hence or otherwise determine whether the method can be applied safely to the four figure tables supplied, in the following cases:

- (i) to find $\log 1.45$ from $\log 1.40$ and $\log 1.50$,
- (ii) to find $\cot 1^\circ 45'$ from $\cot 1^\circ 40'$ and $\cot 1^\circ 50'$.

Question (1914 STEP I Q304)

Give examples to illustrate the utility of the method of reciprocation in geometry.

Question (1941 STEP III Q306)

If

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n,$$

prove that

$$\begin{aligned} \int_{-1}^1 P_m(x) P_n(x) dx &= 0 \text{ if } m \neq n \\ &= \frac{2}{2n+1} \text{ if } m = n. \end{aligned}$$

Question (1924 STEP II Q402)

Find the rationalized form of $x^{1/r} + y^{1/r} + z^{1/r} = 0$ in the cases $r = 3$ and 4 .

Question (1933 STEP II Q408)

Polynomials $f_0(x), f_1(x), f_2(x), \dots$ are defined by the relation

$$f_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n.$$

Prove that

$$\int_{-1}^{+1} f_n(x) f_m(x) dx = 0$$

if $m \neq n$, and that

$$\int_{-1}^{+1} \{f_n(x)\}^2 dx = \frac{2(n!)^2 2^{2n+1}}{2n+1}.$$

Shew that if $\phi(x)$ is any polynomial of degree m ,

$$\phi(x) = \sum_{n=0}^m a_n f_n(x),$$

where

$$a_n = \frac{2n+1}{(n!)^2 2^{2n+1}} \int_{-1}^{+1} \phi(x) f_n(x) dx.$$

Question (1916 STEP III Q506)

If p_n/q_n be the n th convergent to $\sqrt{a^2 + 1}$ when expressed as a continued fraction, prove that

$$\begin{aligned} 2p_n &= q_{n-1} + q_{n+1} \\ \text{and } 2(a^2 + 1)q_n &= p_{n-1} + p_{n+1}. \end{aligned}$$

Question (1934 STEP III Q510)

If $y_r(x)$ satisfies the equation

$$\frac{d}{dx} \left((1 - x^2) \frac{dy}{dx} \right) + r(r+1)y = 0,$$

shew that if $m \neq n$ then

$$\int_{-1}^{+1} y_m(x) y_n(x) dx = 0.$$

Question (1913 STEP II Q611)

Prove the formula for the radius of curvature $\rho = r \frac{dr}{dp}$. At any point of a rectangular hyperbola prove that $3\rho \frac{d^2 p}{ds^2} - 2 \left(\frac{dp}{ds} \right)^2$ is constant.

Question (1925 STEP I Q709)

Evaluate the integrals:

$$\int_0^\infty \frac{x^{p-1}}{1+x+x^2} dx \quad (0 < p < 1), \quad \int_0^\infty \frac{x^2}{\sinh^2 x} dx.$$

Question (1920 STEP III Q705)

Defining the Legendre Polynomial of degree n (positive integral) by the equation

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n,$$

show that

- (a) $P_n(1) = 1, \quad P_n(-1) = (-1)^n.$
- (b) $(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0.$
- (c) $\int_{-1}^{+1} P_n(x)P_m(x)dx = 0$ if $m \neq n$, $= \frac{2}{2n+1}$ if $m = n$.

Question (1920 STEP III Q706)

Define the Weierstrassian Elliptic Function $\wp(u)$ as the sum of a double series and verify that it is doubly periodic. Prove that, if $u + v + w = 0$, then

$$\begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(w) & \wp'(w) \end{vmatrix} = 0.$$

Question (1924 STEP III Q704)

Prove that

$$\frac{3^3}{1.2} + \frac{5^3}{1.2.3} + \frac{7^3}{1.2.3.4} + \cdots = 21e.$$

Question (1923 STEP I Q808)

Show how the number and approximate position of the real roots of an algebraic equation may be determined by means of the properties of a series of Sturm's functions. Show that the Legendre polynomials $P_0(x), P_1(x), \dots, P_n(x)$ have the characteristic property of a series of Sturm's functions and state what information regarding the zeros of $P_n(x)$ can be obtained from this fact.

Question (1923 STEP I Q812)

Prove that, if 2ω is a period of $\wp u$, then

$$\frac{\wp'(u + \omega)}{\wp'u} = - \left\{ \frac{\wp(u/2) - \wp\omega}{\wp u - \wp\omega} \right\}^2,$$

and verify the formula by making $u \rightarrow \omega$.

Question (1924 STEP I Q814)

Prove the addition formula

$$\wp(u + v) = \frac{1}{4} \left(\frac{\wp'u - \wp'v}{\wp u - \wp v} \right)^2 - \wp u - \wp v$$

for the Weierstrassian elliptic function $\wp u$, and deduce a formula for $\wp(2u)$. Shew that if $\wp u$ has primitive periods $2\omega_1, 2\omega_2$, and invariants g_2, g_3 , then $\wp\left(\frac{2\omega_1}{3}\right)$ is a root of the equation

$$48x^4 - 24g_2x^2 - 48g_3x - g_2^2 = 0.$$

Find the other roots of this equation.