Question (1968 STEP III Q5)

The integral

$$I = \int_{x-h}^{x+h} f(u)du$$

is to be approximated by an expression of the form $J = af(x-h)+bf(x)+cf(x+h)$, where a, b and c may depend on h but are independent of the function f and of x. Show that a, b and c may be chosen in such a way that $I = J$ whenever f is a polynomial of sufficiently low degree n, and find the largest n for which this is true. Find values of a, b, c such that $I = J$ whenever $f(u) = p + q\sin u + r\cos u$.

None

Question (1975 STEP III Q7)

Let $f_n(x) = (x^2-1)^n$ and let $\phi_n(x) = \frac{d^n}{dx^n}\{f_n(x)\}$. Use Leibniz' theorem on the differentiation of products to show that

$$\frac{d^r}{dx^r}\{f_n(x)\}$$

vanishes at $x = 1$ and $x = -1$ for all values of $r < n$. Hence show that $\int_{-1}^{1} x^k \phi_n(x)dx = 0$ for all $k < n$, and deduce that if $m \neq n$ then

$$\int_{-1}^{1} \phi_m(x)\phi_n(x)dx = 0.$$

None

Question (1979 STEP III Q3)

The real polynomial $f(x)$ has degree 5. Prove that

$$\int_{-y}^{+y} f(x)dx = \frac{1}{2}y\{f(\lambda y) + f(\mu y) + f(-\lambda y) + f(-\mu y)\}$$

for positive constants λ and μ (independent of y and f) whose squares are the roots of a certain quadratic equation to be determined.

None

Question (1963 STEP IIII Q210)

Show that there is a unique pair of real numbers a, b with the property that

$$\int_{-1}^{+1} P(x)dx = P(a) + P(b)$$

for all polynomials $P(x)$ of degree at most three.

None

Question (1962 STEP II Q109)
The function $L_n(x)$ is defined by

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}),$$

where n is a positive integer or zero. Show that $L_n(x)$ is a polynomial of degree n, that the coefficient of x^n is $(-1)^n$ and that $L_n(0) = n!$. By substituting for $L_n(x)$, but not for $L_m(x)$, and integrating by parts, or otherwise, show that

$$\int_0^\infty L_m(x)L_n(x)e^{-x}dx = \begin{cases} 0 & (n > m \geq 0), \\ (n!)^2 & (m = n). \end{cases}$$

None

Question (1963 STEP II Q301)
Write $f_n(x)$ for the polynomial $d^n/dx^n(x^2-1)^n$. Prove that if $k < n$

$$\int_{-1}^1 x^k f_n(x)dx = 0.$$

Deduce that, if $g_n(x) = d/dx\{(x^2-1)f_n'(x)\}$ and $k < n$,

$$\int_{-1}^1 x^k g_n(x)dx = 0.$$

Hence show that, if λ is the constant such that the coefficient of x^n in $h_n(x) = g_n(x) - \lambda f_n(x)$ vanishes, $h_n(x)$ is identically zero.

None

Question (1955 STEP II Q409)
$I(p,q)$ is defined as

$$\int_0^1 x^p(1-x)^q dx,$$

where p and q are real and non-negative. Show that

$$I(p,q) = I(q,p).$$

Obtain a reduction formula for the integral and state any limitations on the values of p and q necessary. Prove that if p and q are positive integers

$$I(p,q) = p!q!/(p+q+1)!$$

Question (1957 STEP II Q406)
The polynomial $f_n(x)$ is defined as $\dfrac{d^n}{dx^n}(x^2-1)^n$. Prove that all the roots of the equation $f_n(x) = 0$ are real and distinct and lie between ± 1. Prove also that $\int_{-1}^1 f_n(x)f_m(x)dx = 0$ if $m \neq n$, and find its value when $m = n$.

Question (1944 STEP IIII Q307)

If
$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}),$$
show that

(i) $xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0,$

(ii) $\int_0^\infty e^{-x} x^k L_n(x)\,dx = 0$

if k is an integer less than n.

Question (1924 STEP I Q112)

Prove by induction or otherwise that
$$\int_0^\pi \{\sin n\theta / \sin\theta\}\,d\theta = 0 \text{ or } \pi$$
according as n is an even or odd positive integer.

Question (1923 STEP I Q105)

Prove that, if
$$f_n(x) = \frac{1}{2^n.n!}\frac{d^n}{dx^n}\{(x^2-1)^n\},$$
then
$$f_n(1) = 1, \quad f_n(-1) = (-1)^n.$$
Prove also, by integration by parts or otherwise, that, if $\phi(x)$ is a polynomial of degree less than n,
$$\int_{-1}^1 \phi(x) f_n(x) dx = 0,$$
and that
$$\int_{-1}^1 f_n(x) f_m(x) dx = 0, \text{ or } 2/(2n+1),$$
according as $m \neq n$, or $m = n$.

Question (1920 STEP IIII Q204)

Give a discussion of the method of "proportional parts" as applied to interpolation in mathematical tables; and by considering the function

$$F(x) = f(x) - f(a) - \frac{x-a}{b-a}\{f(b) - f(a)\} - C(x-a)(x-b), \text{ or otherwise,}$$

shew that the error in the value of $f(c)$ as calculated from the tabular values given for $x = a, x = b$, is equal to

$$\frac{1}{2}(b-c)(c-a)f''(\gamma)$$

in excess of the true value, where c and γ lie between a and b. Hence or otherwise determine whether the method can be applied safely to the four figure tables supplied, in the following cases:

(i) to find $\log 1.45$ from $\log 1.40$ and $\log 1.50$,

(ii) to find $\cot 1°45'$ from $\cot 1°40'$ and $\cot 1°50'$.

Question (1914 STEP I Q304)

Give examples to illustrate the utility of the method of reciprocation in geometry.

Question (1941 STEP III Q306)

If

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n,$$

prove that

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \text{ if } m \neq n$$

$$= \frac{2}{2n+1} \text{ if } m = n.$$

Question (1924 STEP II Q402)

Find the rationalized form of $x^{1/r} + y^{1/r} + z^{1/r} = 0$ in the cases $r = 3$ and 4.

Question (1933 STEP II Q408)

Polynomials $f_0(x), f_1(x), f_2(x), \ldots$ are defined by the relation

$$f_n(x) = \frac{d^n}{dx^n}(x^2-1)^n.$$

Prove that

$$\int_{-1}^{+1} f_n(x) f_m(x) dx = 0$$

if $m \neq n$, and that

$$\int_{-1}^{+1} \{f_n(x)\}^2 dx = \frac{2(n!)^2 2^{2n+1}}{2n+1}.$$

Shew that if $\phi(x)$ is any polynomial of degree m,

$$\phi(x) = \sum_{n=0}^{m} a_n f_n(x),$$

where

$$a_n = \frac{2n+1}{(n!)^2 2^{2n+1}} \int_{-1}^{+1} \phi(x) f_n(x) dx.$$

Question (1916 STEP III Q506)

If p_n/q_n be the nth convergent to $\sqrt{a^2+1}$ when expressed as a continued fraction, prove that

$$2p_n = q_{n-1} + q_{n+1}$$
$$\text{and} \quad 2(a^2+1)q_n = p_{n-1} + p_{n+1}.$$

Question (1934 STEP III Q510)

If $y_r(x)$ satisfies the equation

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + r(r+1)y = 0,$$

shew that if $m \neq n$ then

$$\int_{-1}^{+1} y_m(x) y_n(x) dx = 0.$$

Question (1913 STEP II Q611)

Prove the formula for the radius of curvature $\rho = r\frac{dr}{dp}$. At any point of a rectangular hyperbola prove that $3\rho\frac{d^2p}{ds^2} - 2\left(\frac{dp}{ds}\right)^2$ is constant.

Question (1925 STEP I Q709)

Evaluate the integrals:

$$\int_0^\infty \frac{x^{p-1}}{1+x+x^2}\,dx \quad (0<p<1), \qquad \int_0^\infty \frac{x^2}{\sinh^2 x}\,dx.$$

Question (1920 STEP III Q705)

Defining the Legendre Polynomial of degree n (positive integral) by the equation

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n,$$

show that

(a) $P_n(1) = 1, \quad P_n(-1) = (-1)^n$.

(b) $(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$.

(c) $\int_{-1}^{+1} P_n(x)P_m(x)dx = 0$ if $m \neq n$, $= \frac{2}{2n+1}$ if $m = n$.

Question (1920 STEP III Q706)

Define the Weierstrassian Elliptic Function $\wp(u)$ as the sum of a double series and verify that it is doubly periodic. Prove that, if $u+v+w=0$, then

$$\begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(w) & \wp'(w) \end{vmatrix} = 0.$$

Question (1924 STEP III Q704)

Prove that

$$\frac{3^3}{1.2} + \frac{5^3}{1.2.3} + \frac{7^3}{1.2.3.4} + \cdots = 21e.$$

Question (1923 STEP I Q808)

Show how the number and approximate position of the real roots of an algebraic equation may be determined by means of the properties of a series of Sturm's functions. Show that the Legendre polynomials $P_0(x), P_1(x), \ldots, P_n(x)$ have the characteristic property of a series of Sturm's functions and state what information regarding the zeros of $P_n(x)$ can be obtained from this fact.

Question (1923 STEP I Q812)

Prove that, if 2ω is a period of $\wp u$, then

$$\frac{\wp'(u+\omega)}{\wp' u} = -\left\{\frac{\wp(u/2) - \wp\omega}{\wp u - \wp\omega}\right\}^2,$$

and verify the formula by making $u \to \omega$.

Question (1924 STEP I Q814)

Prove the addition formula

$$\wp(u+v) = \frac{1}{4}\left(\frac{\wp' u - \wp' v}{\wp u - \wp v}\right)^2 - \wp u - \wp v$$

for the Weierstrassian elliptic function $\wp u$, and deduce a formula for $\wp(2u)$. Shew that if $\wp u$ has primitive periods $2\omega_1, 2\omega_2$, and invariants g_2, g_3, then $\wp\left(\frac{2\omega_1}{3}\right)$ is a root of the equation

$$48x^4 - 24g_2x^2 - 48g_3x - g_2^2 = 0.$$

Find the other roots of this equation.