Question (1972 STEP I Q12)

Show that, if the cubic equation $x^3 - a_1x^2 + a_2x - a_3 = 0$ has roots α, β, γ and if $a_3 \neq 0$, then $\frac{a_2}{a_3} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. Deduce that the equation $x^3 - ax^2 + 11bx - 4b = 0$ cannot have three strictly positive integer roots. Find a value of a such that $x^3 - ax^2 + 96x - 108 = 0$ does have three positive, integer roots.

Question (1977 STEP I Q5)

The roots of the equation $x^3 + ax^2 + bx + c = 0$ are distinct and form a geometric progression. Taken in another order, they form an arithmetic progression. Find b and c in terms of a.

Question (1978 STEP I Q1)

Suppose that a, b and c are real numbers such that the equation

$$x^3 - ax^2 + bx - c = 0$$

has three distinct real roots, which are in geometric progression. Prove that $abc > 0$ and that

$$\left|\frac{a}{c} - 1\right| > 2.$$

Question (1984 STEP II Q6)

The equation $x^3 + ax^2 + bx + c$ $(c \neq 0)$ has three distinct roots which are in geometric progression and whose reciprocals may be rearranged to form an arithmetic progression. Find b and c in terms of a.

Question (1976 STEP III Q1)

Let b and c be real numbers. The cubic equation $x^3 + 3x^2 + bx + c = 0$ has three distinct real roots which are in geometric progression. Show that there are unique values of b and c such that the roots of this equation are integers, and find this equation and its roots.

Question (1959 STEP IIII Q302) (i) If all the roots of the equation $x^3 + px^2 + qx + r^3 = 0$ are positive, show that $p \leq 3r$ and $q \geq 3r^2$.

(ii) The numbers a, b, c are positive, and

$$d = (b + c - a)(c + a - b)(a + b - c).$$

By considering d^2, or otherwise, show that $d \leq abc$.

Question (1954 STEP IIII Q301)

Find the conditions that the roots of the equation

$$x^3 + 3px^2 + 3qx + r = 0$$

should be (i) in arithmetic progression, (ii) in geometric progression, (iii) in harmonic progression.

Question (1946 STEP II Q301)

If a and b are real numbers, show that the equation

$$x^4 + ax^3 + (b-2)x^2 + ax + 1 = 0$$

has four real roots if and only if one of the following two sets of conditions is satisfied:

1. $b \leq 0, a^2 < 16b^2$;
2. $b \geq 8, 4(b-4) < a^2 < \frac{1}{4}b^2$.

Question (1929 STEP I Q104)

Find for what values of a and b the roots of the equation

$$x^4 - 4x^3 + ax^2 + bx - 1 = 0$$

are in arithmetical progression.

Question (1931 STEP I Q104)

Prove that if $\tan\alpha, \tan\beta, \tan\gamma$ are in arithmetic progression, then so are $\cot(\alpha-\beta)$, $\tan\beta$, $\cot(\gamma-\beta)$.

Question (1935 STEP I Q106)

PSP', QSQ' are any two focal chords of a parabola. Shew that the common chord of the circles described on PP', QQ' as diameters passes through the vertex of the parabola.

Question (1920 STEP II Q202)

Form the equation whose roots are the sum and product of the reciprocals of the roots of the equation

$$x^2 + \lambda x + \mu = 0.$$

If the equation thus formed is identical with the original quadratic equation, prove that

$$\lambda^2 = (1 - \lambda)^2$$

and

$$\mu^3 = \mu^2 + 1.$$

Question (1925 STEP II Q201)

Find the condition that the equations

$$ax^2 + 2bx + c = 0, \quad a'x^2 + 2b'x + c' = 0$$

may have a common root. Prove that, if a, c, a', b', c' are given so that $b'^2 > a'c'$, two real values b_1, b_2 of b can be found to ensure a common root; and form the equation whose roots are the other roots of the equations

$$ax^2 + 2b_1x + c = 0, \quad a'x^2 + 2b_2x + c' = 0.$$

Question (1941 STEP II Q202)

The quartic equation

$$4x^4 + \lambda x^3 + 35x^2 + \mu x + 4 = 0$$

has its roots in geometric progression. What real values might be taken for the ratio of the progression?

Question (1926 STEP IIII Q203)

Find the conditions that the roots of

$$x^3 - ax^2 + bx - c = 0$$

shall be (i) in G.P., (ii) in A.P., (iii) in H.P. Show that if the roots are not in A.P. then there are in general three transformations of the form $x = y + \lambda$ such that the transformed cubic in y has its roots in G.P.

Question (1923 STEP I Q303)

State and prove the harmonic properties of a quadrilateral. P is a variable point upon a conic which circumscribes the triangle ABC. AP, BC meet in Q; AB, PC in R. Shew that QR always passes through a fixed point.

Question (1913 STEP I Q405)

Explain how $\sqrt{13}$ can be expanded as a simple continued fraction. Shew that, if p_n/q_n is the nth convergent, $p_4/q_4 = 48/13$; and prove the relations

$$p_{2n+2} = 10p_{2n} - p_{2n-2}, \quad q_{2n+2} = 10q_{2n} - q_{2n-2}.$$

Question (1930 STEP I Q409)

(i) Prove that $x = 2\sin 10^\circ$ is a root of the equation $x^3 - 3x + 1 = 0$, and find the other two roots. (ii) If $c = \cos^2\theta - \frac{1}{3}\cos^3\theta\cos 3\theta + \frac{1}{5}\cos^5\theta\cos 5\theta - \dots$ to infinity, prove that

$$\tan 2c = 2\cot^2\theta.$$

Question (1934 STEP II Q402)

Solve the equation

$$81x^4 + 54x^3 - 189x^2 - 66x + 40 = 0,$$

given that the roots are in arithmetic progression.

Question (1925 STEP II Q504)

Prove that if n is a positive integer,

$$\cos nx - \cos n\theta = 2^{n-1}\prod_{r=0}^{n-1}\left\{\cos x - \cos\left(\theta + \frac{2r\pi}{n}\right)\right\}.$$

Deduce a product for $\sin n\theta$. Also show that

$$\cos\frac{\pi}{n}\cos\frac{2\pi}{n}\dots\cos\frac{(2n-1)\pi}{n} = \frac{(-1)^n - 1}{2^{2n-1}}.$$

Question (1931 STEP III Q503)

(i) Solve

$$\frac{x^2 - a^2}{(x-a)^3} - \frac{x^2 - b^2}{(x-b)^3} + \frac{x^2 - c^2}{(x-c)^3} = 0$$

$$\frac{(x+a)^3}{(x+a)^3} - \frac{(x+b)^3}{(x+b)^3} + \frac{(x+c)^3}{(x+c)^3} = 0$$

for x, where a, b, c are unequal. *[Note: The second equation appears to have typos in the source; it's transcribed as written, but likely intended to be different.]* (ii) Shew that if the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ are in harmonic progression, then $d^3 = 4cde - 8be^2$, and $25ad^2e = (cd - eb)(11eb - cd)$. Verify these conditions in the case of $40x^4 - 22x^3 - 21x^2 + 2x + 1 = 0$ and solve for x.

Question (1923 STEP II Q603)

Prove that the roots of the equation

$$x^3 + 3px^2 + 3qx + r = 0$$

will be in geometrical progression if

$$p^3r = q^3;$$

but will be in harmonical progression if

$$2q^3 = r(3pq - r).$$

Question (1926 STEP II Q606)

Prove that in general three normals can be drawn to a parabola through a given point. ABC is an equilateral triangle inscribed in a parabola of which A is the vertex. The normals at two points P and Q on the parabola meet in B. Prove that the length of PQ is twice the latus-rectum of the parabola and that the orthocentre of the triangle BPQ lies in AC.