Question (1967 STEP II Q7)

The coefficients a_1 and a_2 of the differential equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$$

are real numbers. Write down the general real solution of the equation. It is given that every real solution of this equation is bounded for $x \geq 0$ — that is, if $f(x)$ is a real solution, there exists a constant M such that $|f(x)| \leq M$ for all $x \geq 0$. Show that a_1 and a_2 must be non-negative.

Question (1969 STEP II Q8)

Find a solution of $d^2y/dx^2 = y$ for which $y = 0$ when $x = l$, and $y = a$ when $x = 0$. Assuming a particular integral of the form $x(A\cosh x + B\sinh x)$, or otherwise, solve

$$\frac{d^2y}{dx^2} = y + 2\cosh(l - x),$$

given that $y = 0$ when $x = l$, and $y = a$ when $x = 0$.

Question (1971 STEP II Q1)

If $f(x) = e^{-ax}\sin(bx + c)$, $a > 0$, and $b > 0$, show that the values of x for which $f(x)$ has either a maximum or a minimum form an arithmetic progression with difference π/b. Show further that the values of $f(x)$ at successive maxima form a geometric progression with ratio $e^{-\pi a/b}$. Find the points of inflexion of $f(x)$. Describe a physical problem for which $f(x)$ might be a solution.

Question (1974 STEP II Q4)

A measuring device has an indicator whose position satisfies the equation

$$\frac{d^2x}{dt^2} + x = -2k\frac{dx}{dt}.$$

Initially,

$$x(0) = 1, \left.\frac{dx}{dt}\right|_{t=0} = -k.$$

Find the solution $x(t)$ when $k > 0$, $k \neq 1$. Sketch the graph of $x(t)$ in the two cases $k = 1$, $k = 2$. Show that, for those values of k between 0 and 1 for which $|x(2)| < 10^{-3}$, we have $|x(3)| \geq 10^{-3}$.

Question (1978 STEP II Q2)

In the differential equations

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0, \quad (A)$$
$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x), \quad (B)$$

p and q are constants. Prove that

(i) the sum of any two solutions of (A) is a solution of (A);

(ii) the sum of any solution of (A) and any solution of (B) is a solution of (B).

Find the solution of the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}$$

which vanishes when $x = 0$ and when $x = \log_e 2$.

Question (1982 STEP II Q1)

Show that the second order differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

with b and c constants, can be written as the pair of equations

$$\frac{dp}{dx} - m_1 p = f(x),$$

$$\frac{dy}{dx} - m_2 y = p,$$

where m_1, m_2 are constants to be determined. Hence, or otherwise, find the general solution of

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}$$

Question (1969 STEP III Q14)

In the electric circuit below, the charge Q on the capacitor C is related to the applied electromotive force E by the differential equation

$$L\frac{d^2Q}{dt^2} + 2R\frac{dQ}{dt} + Q/C = E,$$

where $L > R^2C$, and $E(t) = E_0 \cos \omega t$, where E_0 and ω are constants. Show that the current $I(t)(= dQ/dt)$ is ultimately in phase with $E(t)$ if and only if $\omega^2 LC = 1$.

(0,0) to[R, l=R] (3,0) to[C, l=C] (3,-2) to[R, l=R] (0,-2) to[L, l=L] (0,0); (-1,0) to[sV, l=$E\sim$] (-1,-2) to (0,-2);

Question (1976 STEP III Q6)

A commercial process is governed by the equation $\ddot{x} + 3\dot{x} - 4x = 0$. At the first time T that $|x(T)| = 100$ the process must be shut down at once. The total profit made on that run is then T thousand pounds. Unfortunately the initial values $x(0)$, $\dot{x}(0)$ cannot be controlled exactly and all that can be done is to ensure that $|x(0)|, |\dot{x}(0)| \leq 1$. Estimate the minimum value of T. A machine is available which would improve the accuracy with which $x(0)$, $\dot{x}(0)$ are controlled in such a way that $|x(0)|, |\dot{x}(0)| \leq 1/10$ but this would cost an extra S thousand pounds per run. Would you advise the use of such a machine if $S = 1$, if $S = 4$ or if $S = 10$ and why? (Clearly you have not got as much information as you might want in ideal circumstances but you do have enough information to come to a sensible decision.) [$\log_e 10 = 2.30256.$]

Question (1958 STEP II Q110)

Prove that the solution of the differential equation $\frac{dy}{dx} + ay = f(x)$, where a is constant, is $y = y_0 e^{-ax} + \int_0^x f(u)e^{a(u-x)}du$, where $y_0 = y(0)$. Hence, or otherwise, solve $\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = \begin{cases} 1, & (0 < x < 1) \\ 0, & (x > 1), \end{cases}$ where a and b are constants $(a \neq b)$, given that $y = 0$, $dy/dx = 0$ for $x = 0$, and that y and dy/dx are continuous.

Question (1961 STEP II Q108)

Prove, by substitution or otherwise, that the solution of the differential equation $y'' + n^2 y = f(x)$ with the conditions $y(0) = y'(0) = 0$ is

$$y(x) = \int_0^x \frac{1}{n}\sin(x-t)f(t)dt.$$

Solve the problem in the particular case $f(x) = \sin nx$.

Question (1957 STEP II Q110)

Solve the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

with the conditions $y = 2$ and $\dfrac{dy}{dx} = -5$ at $x = 0$. Hence, or otherwise, find u_n, given that

$$u_{n+2} + 6u_{n+1} + 9u_n = 0$$

for $n \geq 0$, and $u_0 = 2, u_1 = -5$.

Question (1945 STEP I Q104)

The sequence $u_0, u_1, u_2, \dots$ is defined by

$$u_0 = 1, \quad u_1 = 2, \quad u_n = 2u_{n-1} - 5u_{n-2} \quad (n = 2, 3, \dots).$$

Obtain the general expression for u_n.

$$u_n = \frac{1}{2} 5^{\frac{1}{2}(n+1)} \sin(n+1)\theta,$$

where θ is the acute angle defined by $\tan\theta = 2$.

Question (1927 STEP IIII Q205)

Defining $\cos x$ and $\sin x$ as solutions of the differential equation $\frac{d^2y}{dx^2} + y = 0$ with suitable given values of y and $\frac{dy}{dx}$ at $x = 0$, and defining $\frac{\pi}{2}$ as the least root of the equation $\cos x = 0$, obtain, without the use of infinite series, the principal properties of $\cos x$ and $\sin x$.