

**Question (1982 STEP I Q14)**

A light spring has natural length  $a$  and is such that when compressed a distance  $x$  it produces a force of magnitude  $kx$ . It joins two particles of masses  $m_1$  and  $m_2$ . The spring is compressed a distance  $b$  and the system released from rest on a smooth table, so that the particles move in a straight line. Find the positions of the particles at time  $t$  later.

None

**Question (1971 STEP II Q15)**

Three linear springs each of modulus  $\lambda$  and natural length  $l$  are connected end to end and lie in a straight line on a smooth horizontal table. At each of the two points where the springs join, a mass  $m$  which is free to move is attached. The two ends of the composite spring are attached to the table, so that in equilibrium the springs are all stretched. If  $x$  and  $y$  denote small displacements of the masses from their equilibrium positions along the line of the springs, show that

$$ml(\ddot{x} + \ddot{y}) + \lambda(x + y) = 0$$

and

$$ml(\ddot{x} - \ddot{y}) + 3\lambda(x - y) = 0.$$

Describe exactly the subsequent motion if, at  $t = 0$ , one of the masses is given a sudden unit velocity towards the second which is itself stationary.

None

**Question (1977 STEP II Q16)**

A bead of mass  $m_1$  can slide freely and without friction on a straight horizontal wire. A second bead of mass  $m_2$  hangs from the first bead by a string of constant length  $l$ . Find the frequency of small oscillations about the equilibrium configuration. [You may assume that the centre of gravity of the two beads does not move horizontally.]

None

**Question (1979 STEP III Q12)**

A vibrating carbon dioxide molecule can be thought of as three particles constrained to move along a line, the outer two particles each of mass 16 units being joined to the central particle of mass 12 units by identical springs. If the displacements of the three particles from their equilibrium positions are  $x_1$ ,  $x_2$  and  $x_3$  ( $x_2$  referring to the central particle), write down the equation of motion for each particle. Show that these equations can be satisfied by two modes of vibration

$$\begin{aligned} \text{(I)} \quad & x_1 = \cos \omega t, \quad x_2 = 0, \quad x_3 = -\cos \omega t \\ \text{and (II)} \quad & x_1 = \cos \Omega t, \quad x_2 = -A \cos \Omega t, \quad x_3 = \cos \Omega t \end{aligned}$$

with suitable choices of  $\omega$ ,  $\Omega$  and  $A$ . Show that the ratio of the frequencies of the two modes is  $\sqrt{\frac{4}{3}}$ .

None

**Question (1984 STEP III Q15)**

An elastic string is held between two fixed supports P, Q which are a distance  $3d$  apart. The tension in the string is proportional to its extension, and is  $k^2md$  when the length of the string is  $3d$ . A bead of mass  $m$  is attached to the string a distance  $d$  from P, and an identical bead is attached a distance  $d$  from Q. Find the equations of motion of small displacements  $\alpha, \beta$  of the beads perpendicular to PQ. Ignore gravity and motion parallel to PQ. Show that  $\alpha + \beta$  and  $\alpha - \beta$  each undergo simple harmonic motion, and find the periods. Describe the motions corresponding to  $\alpha + \beta = 0$ , and  $\alpha - \beta = 0$ . At time  $t = 0$

$$\alpha = \alpha_0, \beta = \beta_0, \frac{d\alpha}{dt} = \frac{d\beta}{dt} = \gamma_0.$$

Show that if

$$\frac{(\alpha_0 + \beta_0)k}{2\gamma_0} = -\tan\left(\frac{\pi}{2\sqrt{3}}\right)$$

then  $\alpha = \beta = 0$  at some subsequent time.

None

**Question (1969 STEP III Q18)**

A particle is suspended from a fixed point by a light spring. If  $c$  is the extension of the spring when the particle hangs in equilibrium, and  $2\pi/\omega$  is the period of small vertical oscillations when equilibrium is disturbed, show that  $c\omega^2 = g$ . From this particle is now suspended a second particle, of the same mass, by a similar spring. The particles are set in motion in a vertical line. Denoting the extensions of the upper and lower springs by  $2c + x$  and  $c + y$  respectively, write down the equations of motion. Show that two periodic motions each of the form

$$x = a \cos \omega t, \quad y = b \cos \omega t$$

are possible, the frequencies being given by

$$(\omega/\omega_0)^2 = \frac{1}{2}(3 \pm \sqrt{5}).$$

Find the corresponding values of  $b/a$ .

None

**Question (1973 STEP III Q14)**

Four equal stretched strings  $X_0X_1, X_1X_2, X_2X_3, X_3X_4$ , each of natural length  $l$ , and modulus of elasticity  $\lambda m$ , lie in a straight line on a smooth horizontal table. The ends  $X_0, X_4$  are fixed, and masses  $m, nm, m$  are attached to the points  $X_1, X_2, X_3$ , respectively. The system performs oscillations along the line of the springs. Determine the equations of motion for the masses in terms of their displacements from their equilibrium positions. Show that if all the masses oscillate with the same period  $2\pi/p$ , then in order to have a non-trivial solution, either  $p^2 = 2\lambda$ , or  $p^2$  satisfies the equation

$$(2\lambda - np^2)(2\lambda - p^2) = 2\lambda^2.$$

None

**Question (1959 STEP III Q108)**

A particle  $Q$  of mass  $2m$  is attached to one end of a light elastic string  $PQ$  of length  $2a$  and modulus of elasticity  $\lambda$ ; a particle  $R$  of mass  $3m$  is attached to the mid-point of the string. The system is then hung in equilibrium from a fixed point  $P$ . The particle  $Q$  is given a small downward impulse  $\epsilon\sqrt{\frac{m\lambda}{a}}$ . After time  $t$  the ensuing displacements of  $Q$ ,  $R$  from the equilibrium position are  $x$ ,  $y$ , respectively. Prove that  $\ddot{x} = -3\omega^2(x - y)$ ,  $\ddot{y} = 2\omega^2(x - 2y)$ , where  $\omega = \sqrt{\frac{\lambda}{6am}}$ . Verify that  $x = \epsilon\left(\frac{3\sqrt{6}}{10}\sin\omega t + \frac{1}{5}\sin\sqrt{6}\omega t\right)$  satisfies the initial conditions. Deduce that this is the correct solution for  $x$ , by finding a similar formula for  $y$ , which, together with that for  $x$ , satisfies the equations of motion and the initial conditions. Is the motion periodic?

None

**Question (1963 STEP II Q210)**

Two particles, each of mass  $m$ , hang at the ends  $A$ ,  $B$  of two light inextensible strings, each of length  $a$ , the other ends of which are fixed at the same level at a distance  $b$  apart. The particles are joined by a light spring of natural length  $b$  and modulus  $\lambda$  and initially the system is at rest in its equilibrium position. The particle at  $A$  is then struck by an impulse  $I$  directed towards  $B$ . In the subsequent motion the angles  $\theta$ ,  $\phi$  which the strings make with the vertical (measured in the same sense) remain small. Show that

$$\theta + \phi = -\frac{(g/a)(\theta + \phi)}{x}$$

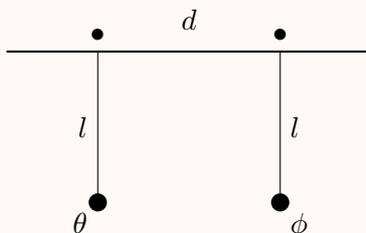
$$\theta - \phi = -\frac{(g/a)(1 + \epsilon)(\theta - \phi)}{x}$$

where  $\epsilon = 2\lambda a/(mga)$ , and hence find  $\theta$  and  $\phi$  as functions of  $t$ . Defend the statement that, if  $\epsilon$  is small, the motion can be described as the repeated transfer, from particle  $A$  to particle and  $B$  back, of an oscillatory motion, with a repetition time approximately  $4\pi\sqrt{(a/g)\epsilon}$ .

None

**Question (1962 STEP III Q105)**

Two identical simple pendulums each of mass  $M$  and length  $l$ , suspended from the same horizontal plane, are connected by a light straight spring (which is both inextensible and extensible) of natural length  $d$  and modulus of elasticity  $\lambda$ , as shown in the figure. The system is released from rest with the pendulums coplanar and  $\theta$  and  $\phi$  small. Prove that the quantities  $(\theta + \phi)$  and  $(\theta - \phi)$  vary periodically with time, and find their approximate periods.



None

**Question (1963 STEP III Q107)**

The elastic strings  $AB$ ,  $BC$  have unstretched lengths  $l$  and moduli of elasticity  $3\lambda mg$  and  $2\lambda mg$  respectively.  $A$  is attached to a fixed support and particles of mass  $m$  are attached at  $B$  and  $C$  and the system hangs in equilibrium vertically. The particles at  $B$  and  $C$  are now displaced vertically downwards through distances  $x_0$  and  $y_0$  respectively from their equilibrium positions and are then released. If the subsequent displacements of the particles from their equilibrium positions are  $x$  and  $y$ , show that  $x + 2y$  and  $2x - y$  vary harmonically with time and find their periods. If  $y_0 = 2x_0$ , find expressions for  $x$  and  $y$  as functions of the time. (It may be assumed that  $x_0$  and  $y_0$  are so small that neither string ever becomes slack.)

None

**Question (1964 STEP III Q209)**

Two similar simple pendulums of length  $l$  are suspended at the same height. They have light bobs attached to the opposite ends of a light inextensible string also of length  $l$ , so that they are pulled together, and the pendulums make a small angle  $s$  with the vertical. The pendulums are displaced at right angles to the original plane of the system through angles small compared with  $s$ . Assuming that all the tensions maintain their original values to the degree of approximation necessary, show that the subsequent displacements of the pendulums can be represented by the sum and difference of two harmonic oscillations with slightly different frequencies. Deduce that if only one pendulum is displaced, the motion is concentrated in the other after a time approximately  $(n/\alpha)\sqrt{l/g}$ . Describe the motion.

None

**Question (1963 STEP III Q304)**

Three springs of unit length and modulus  $M$  are joined together end to end and restricted to lie on a horizontal line. Two masses  $m$  are fixed to the junctions and the outer ends are held fixed. By taking the coordinates  $x_1$  and  $x_2$  to represent the displacements of the two masses from their respective positions of equilibrium, show that two simple harmonic motions are possible, in which either  $x_1 + x_2$  or  $x_1 - x_2$  is zero. What is the ratio of their periods? If the masses are released from rest at arbitrary values of  $x_1$  and  $x_2$ , show that in general at no later time are both particles at rest.

None

**Question (1955 STEP III Q204)**

A uniform rigid wire  $ABC$  consisting of a straight section  $AB$  of length  $2l$  at right angles to a straight section  $BC$  of length  $4l$  is freely suspended at  $A$ . Show that in the position of stable equilibrium  $AB$  makes an angle  $\tan^{-1} 4/5$  with the downward vertical. If the wire makes small oscillations in the vertical plane about the position of equilibrium, find the length of the equivalent simple pendulum.

**Question (1946 STEP III Q110)**

A particle of unit mass moves in a plane under a force with components

$$(-ax - hy, -hx - by)$$

referred to rectangular axes  $Ox, Oy$ , where  $a, b$  and  $h$  are constants such that  $a > 0, b > 0, ab > h^2$ . Show that there are two straight lines through  $O$  along which the particle can move in simple harmonic motion, and that these lines are orthogonal. Determine these lines and the corresponding periods of oscillation for the case  $a = 13, b = 7, h = 3\sqrt{3}$ .

**Question (1946 STEP II Q211)**

A short train consists of an engine of mass  $M$  coupled to a single coach of mass  $m$  whose bearings are smooth. Between the engine and the coach there are two pairs of spring buffers of negligible mass, one pair being on the engine and the other on the coach. The coupling is such that, with the train at rest and the coupling taut, the buffers on the engine are just in contact with the corresponding buffers on the coach but none of the buffers are compressed. Each buffer has a compliance  $C$ , compliance being the ratio of compression to force producing compression. When the train is in steady motion with uniform velocity along a straight track, brakes are applied, but only to the engine. The braking force is such that it would produce a retardation  $f$ , if there were no coach. Prove that, after application of the brakes, the separation between engine and coach oscillates with frequency

$$\frac{1}{2\pi} \sqrt{\frac{M+m}{MmC}}.$$

Prove further that, if this oscillation is damped out, the effect of the retardation is to reduce the separation between engine and coach by

$$\frac{MmCf}{M+m}.$$

**Question (1946 STEP II Q310)**

Two particles of masses  $m, m'$  are attached to the middle point  $A$  and to the end point  $A'$  of a light inextensible string  $OAA'$  of length  $2l$ . The end  $O$  is fixed and the system executes a small oscillation under gravity in a vertical plane through  $O$ . If  $x, x'$  are the horizontal distances of the particles from the vertical line through  $O$  at time  $t$ , and  $n^2 = g/l$ , prove that

$$m \frac{d^2x}{dt^2} + (m + 2m')n^2x - m'n^2x' = 0,$$

$$\frac{d^2x'}{dt^2} + n^2x' - n^2x = 0,$$

and hence show that if  $m = 3m'$ , a motion is possible in which  $x + x' = 0$ .

**Question (1946 STEP III Q105)**

A point is moving with simple harmonic motion, of period  $2\pi/n$  and amplitude  $a$ , in a straight line. If at any instant the distance of the point from its mean position is  $x$ , show that the speed of the point is  $n\sqrt{a^2 - x^2}$ . A mass  $m$  hangs at rest at the lower end of a light elastic string, of unstretched length  $l$  and modulus of elasticity  $\lambda$ . A second mass  $m$ , moving vertically upwards with velocity  $U$ , impinges on the first mass and coalesces with it. Show that in the subsequent motion the string remains taut provided that  $\lambda U^2 < 6mlg^2$ .

**Question (1946 STEP III Q110)**

Two particles,  $A$ ,  $B$ , of masses  $m_1$ ,  $m_2$  respectively, are connected by a light spiral spring, which obeys Hooke's law, and move on a smooth horizontal table along the line of the spring (supposed to remain straight). Write down the equations of motion of the masses in terms of their displacements  $x_1$ ,  $x_2$  and deduce that

1. the centre of gravity  $G$  of the system moves with constant velocity;
2. the distance  $x_2 - x_1$  between  $A$  and  $B$  varies harmonically;
3. the distance of either mass from  $G$  varies harmonically with the same period as  $x_2 - x_1$ .

**Question (1945 STEP III Q208)**

A particle of mass  $m$  is attached by two elastic strings of different moduli of elasticity to two points  $A$ ,  $B$  of a horizontal table. The unstretched lengths of the strings are  $a$ ,  $b$  and the stretched lengths in the equilibrium position  $a'$ ,  $b'$ . If the periods of small oscillations in the directions along  $AB$  and perpendicular to  $AB$  are  $2\pi/n_1$  and  $2\pi/n_2$  respectively, shew that

$$n_1^2 = \frac{1}{m} \left( \frac{\lambda_1}{a} + \frac{\lambda_2}{b} \right) \quad \text{This seems to be missing from the OCR, I will transcribe what is there.}$$

shew that

$$n_2^2 = \frac{1}{m} \left( \frac{1}{a'} + \frac{1}{b'} \right) \quad \text{This seems to be missing from the OCR, I will transcribe what is there.}$$

$$\begin{aligned} n_1^2 &= \dots \quad \frac{1}{a'} + \frac{1}{b'} \\ n_2^2 &= \dots \quad \frac{1}{a' - a} + \frac{1}{b' - b} \\ n_2^{-2} &= \frac{1}{a'} + \frac{1}{b'} \\ n_1^2 &= \frac{1}{a' - a} + \frac{1}{b' - b} \end{aligned}$$

**Question (1944 STEP III Q310)**

A light inelastic string  $ABC$ , of length  $2a$ , has a particle of mass  $m$  attached at its mid-point  $B$ , and a second particle of mass  $m$  is attached to the end  $C$ . The end  $A$  is fixed, and the particle at  $C$  is constrained to move, without friction, on the vertical line through  $A$ . The particle at  $B$  moves in a vertical plane through  $A$ . Prove that, if the system executes small oscillations about the position of stable equilibrium, the length of the equivalent simple pendulum is  $a/3$ . Prove also that, if the amplitude of the motion of  $AB$  is a small angle  $\alpha$ , the tensions in the two parts of the string when  $B$  is vertically below  $A$  are  $mg(2 + 9\alpha^2)$  and  $mg(1 + 6\alpha^2)$ .

**Question (1945 STEP III Q305)**

A particle of mass  $m$  moves on a straight line under a force  $mn^2r$  towards a fixed point  $O$  of the line, where  $r$  denotes distance from  $O$ . Prove that the motion is periodic, with period  $2\pi/n$ . A particle of mass  $m$  is attached to the mid-point of a light uniform elastic string, of natural length  $2a$ . When the ends of the string are attached to fixed points at the same level and at a distance  $2a$  apart, and the particle hangs in equilibrium, the stretched length of the string is  $2b$ . Prove that the period of a small vertical oscillation about the position of equilibrium is  $2\pi/n$ , where

$$n^2 = \frac{b^2 + ba + a^2}{b^2 \sqrt{(b^2 - a^2)}} g.$$

**Question (1946 STEP III Q409)**

A uniform heavy bar of length  $2l$  hangs in equilibrium under gravity by means of two equal crossed strings that are attached to its ends and to two points distance  $2l$  apart at the same horizontal level at a height  $2h$  above the beam. If the motion of the bar is restricted to the vertical plane through the points of suspension, show that, if  $h > l$ , the period of small oscillations about the equilibrium position is

$$2\pi \sqrt{\frac{2h(l^2 + 3h^2)}{3(h^2 - l^2)g}}.$$

**Question (1920 STEP I Q110)**

A particle rests on a smooth horizontal table and is constrained by two springs, attached to fixed points in the plane of the table, whose tensions are  $\mu_1$  and  $\mu_2$  times their lengths. It is started in motion in any manner so as to remain on the table and left to itself. Show that the projection of the motion on any direction is simple harmonic with the same period. What is the most general form of the path of the particle?

**Question (1917 STEP I Q115)**

Two particles can move in the same straight line in a field of force per unit mass directed towards a point in that line and varying as the distance from that point. Shew that consecutive impacts between the particles take place at equal intervals of time and at one or other of two points in the line, and that the greatest distances between the particles during these intervals form a geometrical progression of ratio  $e$ , where  $e$  is the coefficient of restitution between the particles.

**Question (1918 STEP I Q114)**

A mass  $M$  suspended at the end of a vertical spring oscillates harmonically with amplitude  $a$ . At the moment of the greatest velocity of  $M$  a mass  $m$  at rest is placed upon it and remains upon it; shew that the resulting amplitude of oscillation is  $\sqrt{\left(\mu^2 \frac{M}{M+m} + \frac{m^2}{(M+m)^2}\right) a^2}$ , where  $\mu$  is the mass which suspended alone at the end of the spring stretches it unit distance.

**Question (1926 STEP I Q112)**

Find the work done in stretching an elastic string. A particle of mass  $m$  lies upon a smooth horizontal table and is attached to three points upon the table, at the vertices of an equilateral triangle of side  $2a$ , by means of three strings of natural lengths  $l, l'$  and  $l'$  and of moduli  $\lambda, \lambda'$  and  $\lambda'$  respectively. Shew that if the particle can rest in equilibrium at the centre of the triangle, then

$$2a(\lambda/l - \lambda'/l') = (\lambda - \lambda')\sqrt{3}.$$

Find also the period of a small oscillation of the particle in the line of the string of natural length  $l$ .

**Question (1929 STEP I Q110)**

A particle of mass  $m$  is attached by a light spring to a fixed point on a smooth horizontal board of mass  $M$  which can slide in a prescribed direction on a horizontal plane. The spring is capable of longitudinal extension and compression in the prescribed direction. Shew that when motion takes place the number of oscillations per second is  $\sqrt{1 + m/M}$  of what it would be if the board were fixed.

**Question (1937 STEP I Q109)**

A small ring of mass  $m$  slides on a smooth wire in the form of the parabola  $y^2 = 4ax$ , the  $x$ -axis being the downward vertical. It is connected to the focus by a light spring of natural length  $a$ , and rests in equilibrium at the point  $(a\lambda_0^2, 2a\lambda_0)$ . Show that the modulus of the spring is  $mg\lambda_0^{-2}$ . Show further that, if slightly displaced, the spring vibrates with period approximately

$$2\pi\sqrt{\frac{a(\lambda_0^2 + 1)}{g}}.$$

**Question (1938 STEP I Q108)**

The ends of a light spring of natural length  $2a$  and modulus  $\lambda$  are fixed at points  $A, B$  on a smooth horizontal table at distance  $4a$  apart, and a particle of mass  $m$  is fixed to the mid-point of the spring. Write down the equation of motion of the particle along the horizontal perpendicular bisector of  $AB$ , and, by integrating this equation, prove that energy is conserved in the motion. Prove that, for oscillations of small amplitude, the period is approximately  $2\pi\sqrt{\left(\frac{ma}{\lambda}\right)}$ .

**Question (1939 STEP I Q109)**

A small ring slides on a smooth circular wire of radius  $a$  fixed in a vertical plane, and is connected to the highest point of the wire by a light spring of natural length  $a$ . Prove that, in any motion in which the spring remains taut, the energy is constant.

If the ring is in equilibrium at one end of the horizontal diameter, and is then slightly disturbed, prove that it will oscillate with period  $2\pi/n$ , where

$$n^2 = \frac{\sqrt{2} + 1}{2} \frac{g}{a}.$$

**Question (1921 STEP I Q110)**

Define Simple Harmonic Motion, and establish its chief properties. Discuss the result of compounding simple harmonic motions (1) in the same straight line, (2) in perpendicular straight lines, the periods being equal, (3) in perpendicular straight lines, one period being twice the other.

**Question (1930 STEP I Q107)**

A mass  $M$  is hung from a light spring of natural length  $l_1$  and modulus of elasticity  $\lambda_1$  and the other end of the spring is hung from a second light spring ( $l_2, \lambda_2$ ), of which the upper end is fixed. Find the equilibrium configuration, and shew that the period of small vertical oscillations is

$$2\pi\sqrt{M\left(\frac{l_1}{\lambda_1} + \frac{l_2}{\lambda_2}\right)}.$$

**Question (1938 STEP I Q110)**

A light inextensible string of length  $2l$  is fastened at one end to a fixed point; it carries a mass  $m$  at the mid-point and a mass  $2m$  at the lower end. The system is slightly disturbed from rest so that the masses move in the same vertical plane. If the horizontal displacements of the upper and lower masses are  $x_1$  and  $x_2$  respectively (both being measured in the same direction) show that the equations of motion of the masses are

$$\begin{aligned}\frac{d^2x_1}{dt^2} + n^2(5x_1 - 2x_2) &= 0, \\ \frac{d^2x_2}{dt^2} + n^2(x_2 - x_1) &= 0,\end{aligned}$$

where  $n^2 = g/l$ , and terms of higher order are neglected. Show that a state of oscillation is possible in which the masses execute simple harmonic motions of the same period, with  $x_1 = x_2(\sqrt{6} - 2)$ , and find the period of oscillation.

**Question (1940 STEP I Q109)**

Two light spiral springs, OA, AB, are joined together at A, and particles of equal mass are fastened to the compound spring at A, B respectively; the end O is fixed at a point of a smooth horizontal table. Throughout the movement of the system O, A, B remain in a fixed horizontal straight line, with A between O and B. If the masses oscillate so that the displacements of A, B along OAB at any instant are  $x_1, x_2$  respectively, obtain the equations of motion

$$\begin{aligned}\ddot{x}_1 + q^2x_1 - p^2x_2 &= 0 \\ \ddot{x}_2 - p^2x_1 + p^2x_2 &= 0,\end{aligned}$$

where  $2\pi/p$  is the period of oscillation of the mass B when A is held fixed, and  $2\pi/q$  is the period of oscillation of the mass A when B is held fixed with OB equal to the unstretched length of the combined spring.

For the case in which  $q^2 = \frac{5}{2}p^2$  show that a solution of these equations can be obtained in which  $x_1 = H \cos \sqrt{\frac{1}{3}}pt$ ,  $x_2 = K \cos \sqrt{\frac{1}{3}}pt$ , and find the ratio  $H/K$ .

Show also that a second solution  $x_1 = H' \cos \sqrt{\frac{5}{3}}pt$ ,  $x_2 = K' \cos \sqrt{\frac{5}{3}}pt$  exists.

**Question (1922 STEP I Q207)**

Two equal particles A, B are attached to the ends of a spring which is held by its ends vertically and unstretched, A being uppermost. B is released, and at the moment at which it first comes to rest A is also released. Describe fully the subsequent motion, and shew that B comes to rest again once.

**Question (1924 STEP I Q209)**

A light elastic string of unstretched length  $l$  hangs vertically supporting a mass  $m$  and is extended by a length  $b$ . A mass  $m'$  is taken from  $m$  and the remaining mass is set free to oscillate. Find the greatest value of  $m'$  such that the string always remains taut in the subsequent motion. As in the previous case, a mass  $m_1 (< m')$  is removed from  $m$  while hanging at rest. At the top of the ensuing oscillation  $m_1$  is added again and at the bottom of the next oscillation  $m_1$  is removed again. This process of adding and subtracting  $m_1$  is repeated  $n$  times, after which the string just reaches its unstretched length as the mass rises to its highest point. Determine  $m_1$ .

**Question (1926 STEP I Q209)**

A particle of mass  $m$  is moving in the axis of  $x$  under a central force  $\mu mx$  to the origin. When  $t = 2$  seconds, it passes through the origin, and when  $t = 4$  seconds, its velocity is 4 feet per second. Determine the motion and shew that, if the complete period is 16 seconds, the semi-amplitude of the path is  $\frac{32\sqrt{2}}{\pi}$  feet.

**Question (1927 STEP I Q210)**

Two particles  $A, B$ , each of mass  $m$ , are attached to the ends of a light rod of length  $a$ . The rod is horizontal and instantaneously at rest, when  $A$  receives an upward vertical impulse  $mv$ . Prove that, in the subsequent motion, the vertical component of  $B$ 's velocity will always be downwards if  $v^2/2ga$  is less than the least positive root of the equation

$$x \sin(x + \sqrt{x^2 - 1}) = 1.$$

**Question (1931 STEP I Q210)**

A particle of mass  $m$  is attached to the four corners of a square, whose diagonal is of length  $2a$ , by four elastic strings each of length  $a$ , stretched under a tension  $P$ . Determine the period of a small oscillation of the particle for displacements normal to the plane of the square, and shew that for displacements along a diagonal of the square the period is

$$2\pi \left( \frac{ma}{4P + 2\lambda} \right)^{\frac{1}{2}},$$

where  $\lambda$  is the tension in a string when its length is twice the unstretched length.

**Question (1935 STEP I Q207)**

Explain what is meant by simple harmonic motion. A smooth light pulley is suspended from a fixed point by a spring of natural length  $l$  and modulus of elasticity  $\lambda$ . If masses  $m_1$  and  $m_2$  hang over the pulley, show that the pulley executes a simple harmonic motion about a point whose depth below the point of suspension is  $l \left\{ 1 + \frac{4m_1 m_2 g}{m_1 + m_2 \lambda} \right\}$ .

**Question (1932 STEP III Q209)**

Each of three particles  $A, B, C$  has a mass  $m$ , and  $A$  is joined to  $B$ , and  $B$  to  $C$  by similar light springs of natural length  $a$ . The particles move in a straight line under no forces save the tensions of the springs. Shew that if the lengths of  $AB, BC$  respectively at time  $t$  are denoted by  $a + x, a + y$  respectively, then

$$\frac{d^2u}{dt^2} + n^2u = 0, \quad \frac{d^2v}{dt^2} + 3n^2v = 0,$$

where  $u = x + y, v = x - y$ , and  $amn^2$  is the tension required to double the length of either spring. Hence determine the length of  $AB$  at any time if the system, originally at rest with the springs unstretched, is set in motion by an impulse  $I$  on the particle  $C$  in the direction  $AC$ .

**Question (1935 STEP III Q210)**

Two particles of masses  $m$  and  $m'$  are attached to the ends of a spring of natural length  $l$  and modulus  $\lambda$ . They fall with the spring at its natural length and vertical so that  $m'$  strikes a horizontal inelastic table when both masses are moving vertically downwards with velocity  $V$ . Shew that if  $m'$  leaves the table it does so when the velocity of  $m$  is  $V_1$ , given by

$$mV_1^2 = mV^2 - m'(m' + 2m)lg^2/\lambda,$$

and that in the subsequent motion the extension of the spring varies harmonically in the period  $2\pi\sqrt{\{mm'l/\lambda(m+m')\}}$ , the maximum extension being

$$\left( \frac{mm'l\{V^2 - m'lg^2/\lambda\}}{\lambda(m+m')} \right)^{\frac{1}{2}}.$$

**Question (1938 STEP III Q211)**

A rod of mass  $M$  is free to rotate in a vertical plane about a fixed point  $O$ . The moment of inertia of the rod about  $O$  is  $I$ , and the distance of the centre of gravity from  $O$  is  $h$ . When the rod makes an angle  $\alpha$  with the downward vertical, its angular velocity is  $\omega$ . Determine the horizontal and vertical components of the reaction on the hinge when the rod makes an angle  $\theta$  with the downward vertical. Shew that these two components cannot vanish simultaneously in a position in which the rod is not vertical, unless  $I = Mh^2$  and the angular velocity  $\omega$  satisfies the inequalities

$$\frac{2g \cos \alpha}{h} < \omega^2 < \frac{g(2 \cos \alpha + 3)}{h}.$$

If the two components vanish simultaneously when the rod is vertical, prove that

$$\omega^2 = \frac{g}{h} \left[ \frac{4Mh^2 \cos^2 \frac{\alpha}{2}}{I} + 1 \right].$$

**Question (1942 STEP III Q210)**

A uniform rod of mass  $M$  and length  $2l$  is freely pivoted about its centre so that it can rotate in a vertical plane under gravity. A particle of mass  $m$  is attached by a string of length  $a$  to one end of the rod. It is found that it is possible for the system to oscillate about its equilibrium position so that the rod and string remain in the same plane and make equal and opposite angles with the vertical. Shew that  $Ml = 6(a - l)m$ , and find the period of the oscillation.

**Question (1920 STEP II Q310)**

A particle is moving in a straight line under a force to a fixed point in the line proportional to the distance from the point. Prove that the motion is simple harmonic and find the period. Two light elastic strings of natural lengths  $l, l'$  and moduli  $E, E'$  respectively are knotted together to form one string, one end of which is fixed while the other is attached to a particle of mass  $m$  which oscillates freely in a vertical line under the action of gravity and the tension of the string. Prove that the period of an oscillation is the same as that of a simple pendulum of length  $mg(l/E + l'/E')$ .

**Question (1939 STEP III Q309)**

A particle of mass  $m$  is attached to one end  $B$  of a light elastic string  $AB$ , the other end  $A$  being fixed. When the particle hangs in equilibrium the length of the string exceeds the unstretched length by  $a$ . Prove that if the particle executes small vertical oscillations about the position of equilibrium, the period is the same as that of a simple pendulum of length  $a$ .

A similar string  $BC$ , carrying a particle of mass  $\frac{1}{2}m$  at  $C$ , is now attached to the first particle at  $B$ , and the system hangs in equilibrium from  $A$ . If now the particles oscillate vertically, the downward displacements of  $B$  and  $C$  from their equilibrium positions being  $x$  and  $y$ , establish the equations of motion

$$\ddot{x} = n^2(y - 2x), \quad \ddot{y} = n^2(x - y),$$

where  $n^2 = g/a$ . It is assumed that the strings remain taut.

Prove that a motion is possible in which  $x/2 = y/3$  throughout.

**Question (1919 STEP III Q310)**

A particle of mass  $m$ , lying on a smooth horizontal table, is attached to two elastic strings whose natural lengths are  $l$  and  $l'$  and moduli  $\lambda$  and  $\lambda'$  respectively. The other ends of the strings are fixed to two points on the table at a distance apart greater than  $l + l'$ . Shew that if the particle vibrates in the line of the strings, its period will be

$$2\pi \sqrt{\frac{m}{\frac{\lambda}{l} + \frac{\lambda'}{l'}}}.$$

**Question (1940 STEP III Q309)**

A uniform rod of mass  $m$  and length  $2a$  is supported horizontally by two elastic strings, each of natural length  $l$  and modulus of elasticity  $\lambda$ , which are attached to a fixed point vertically above the middle of the rod. In equilibrium the strings make an angle  $\theta$  with the vertical. Show that the period of small oscillations in which the rod remains horizontal is  $2\pi/n$ , where

$$n^2 = \frac{g}{a} \frac{a - l \sin^3 \theta}{a - l \sin \theta} \tan \theta.$$

**Question (1942 STEP III Q306)**

A flywheel of mass 80 lb. is suspended with its axis vertical by three vertical cords placed equidistant from the axis and at the corners of an equilateral triangle of side 1 ft., and each rope is 5 ft. long.

1. What work is needed to rotate the wheel until the three cords just touch one another?
2. If the period of small rotational oscillations of the flywheel about its axis when so suspended is 2 sec., find its radius of gyration.

**Question (1942 STEP III Q310)**

A light uniform rod of length  $2l$  is freely suspended from one end  $A$  and carries a concentrated mass  $m$  at the other end  $B$ . A light spring, producing a force  $\frac{16mg}{l}$  per unit extension or compression, is attached to the rod at its mid-point and lies horizontally so that the rod is vertical when in equilibrium. Find the amplitude and period of small oscillations of the end  $B$  (i) when it is given a small displacement  $a$  in a direction perpendicular to the spring and then set free; and (ii) when it is projected with a small velocity  $v$  in a direction parallel to the spring. Sketch the projection on a horizontal plane of the path of  $B$  when these two oscillations are superposed, assuming them to begin simultaneously.

**Question (1942 STEP I Q410)**

A uniform rod  $AB$  of length  $2a$  can turn without friction about the end  $A$  in a vertical plane. A light elastic string of natural length  $a$  connects the end  $B$  to the point  $C$  vertically above  $A$  such that the length of  $AC$  is  $2a$ . When the system is in equilibrium  $ABC$  is an equilateral triangle. Prove that the period of small oscillations about this equilibrium position is  $2\pi\sqrt{\frac{8a}{3g}}$ .

**Question (1927 STEP II Q409)**

A sphere rolls or slides on a fixed parabolic wire, always touching it at two points. Prove that the centre of the sphere describes part of an equal parabola.

**Question (1926 STEP III Q404)**

Two masses  $M$  and  $m$ , connected by a light spring obeying Hooke's law, fall in a vertical line with the spring unstressed until  $M$  strikes an inelastic horizontal table. Prove that  $M$  will after an interval rise from the table if the distance through which  $M$  has fallen exceeds  $l(1 + \frac{M}{2m})$ , where  $l$  is the extension that would be produced in the spring by a force equal to the weight of  $M$ .

**Question (1933 STEP I Q510)**

A particle of mass  $3m$  is suspended by a light inextensible string of length  $l$  from a body of mass  $m$  which can move freely on a horizontal rail. If the lower particle is released from rest with the string taut and inclined at a small angle  $\alpha$  to the vertical, determine the amplitudes and the periods of the resulting small oscillations.

**Question (1915 STEP III Q512)**

Two particles of equal mass are attached to the ends of a light rod. The rod can turn freely about a point  $O$  at distances  $l$  and  $l'$  respectively from the ends. Prove that, if the rod rotate about a vertical axis through  $O$  with angular velocity  $\omega$ , it is inclined to the vertical at an angle

$$\cos^{-1} \left\{ \frac{(l \sim l')g}{\omega^2(l^2 + l'^2)} \right\}^{\frac{1}{2}}.$$

**Question (1917 STEP III Q512)**

Two equal light rods of length  $l$  are jointed freely to each other and have particles of equal weight attached to their free ends. The rods are placed symmetrically across a smooth circular cylinder of radius  $a$  with its axis horizontal so as to touch the cylinder and with the joint vertically above the axis. The system is released from rest when  $2\alpha$  is the angle between the rods. Prove that if  $2\beta$  is the angle between the rods when the system next comes to rest

$$\sin \alpha \sin \beta \tan \frac{1}{2}(\alpha + \beta) = a/l.$$

**Question (1922 STEP III Q506)**

Two masses  $m_1$  and  $m_2$  are connected by a light spring and placed on a smooth horizontal table. When  $m_2$  is held fixed,  $m_1$  makes  $n$  complete vibrations per second. Show that if  $m_1$  is held fixed,  $m_2$  will make  $n\sqrt{m_1/m_2}$  vibrations per second, and if both are free, they will make  $n\sqrt{(m_1 + m_2)/m_2}$  vibrations per second, the vibrations in all cases being in the line of the spring.

**Question (1927 STEP III Q508)**

Two light elastic strings are fastened to a particle of mass  $m$  and their other ends to fixed points so that the strings are taut. The modulus of each is  $\lambda$ , the tension  $T$  and the lengths  $a, b$ . Show that the period of an oscillation along the line of the strings is

$$2\pi\{mab/(T + \lambda)(a + b)\}^{\frac{1}{2}}.$$

**Question (1920 STEP I Q710)**

Show how Lagrange's equations of motion may be used to determine the small oscillations of a dynamical system with a finite number of degrees of freedom. A uniform rod of length  $2a$  is hung horizontally by means of two vertical strings from two fixed points at distances  $l$  and  $l'$  above the ends of the rod. Show that the three periods of the normal oscillations are

$$2\pi\sqrt{\left\{g\left(\frac{1}{l} + \frac{1}{l'}\right) \pm g\sqrt{\left(\frac{1}{l} + \frac{1}{l'}\right)^2 + \frac{12}{a^2}\left(\frac{1}{l} - \frac{1}{l'}\right)^2}\right\}^{-1}} \quad \text{and} \quad 2\pi\sqrt{\frac{1}{g}\left(\frac{1}{l} + \frac{1}{l'}\right)^{-1}}.$$

**Question (1914 STEP III Q710)**

Show that motion in a straight line under a restoring force proportional to the displacement is the projection on the line of a uniform circular motion. Two light elastic strings are fastened to a particle of mass  $m$  and their other ends fastened to fixed points so that the strings are stretched. The modulus of each is  $\lambda$ , their tension  $T$ , and in equilibrium their stretched lengths are  $a, b$ . If the particle is slightly displaced along the line of the strings, show that the period of a small oscillation is

$$2\pi\sqrt{\frac{mab}{(T + \lambda)(a + b)}}.$$

**Question (1924 STEP III Q714)**

A mass  $m$  is suspended from a spring causing an extension  $a$ . If a mass  $M$  is added to  $m$  find the periodic time of the ensuing motion, and the amplitude of the oscillation.

**Question (1925 STEP III Q713)**

Show that the mutual potential energy of two small magnets of moments  $M, M'$  is

$$MM'(\cos \epsilon - 3 \cos \theta \cos \theta')/r^3,$$

where  $r$  is the distance apart of their centres,  $\epsilon$  the angle between their axes and  $\theta, \theta'$  the angles made by their axes with the line of centres. Three small magnets of equal moment  $M$  can rotate in a plane about their centres, which are fixed at equal intervals  $a$  along a straight line. Find the periods of the normal oscillations about the position of equilibrium in which the axes all point in the same sense along the line of centres.

**Question (1913 STEP II Q810)**

A light string of length  $6l$  is stretched between two fixed points with tension  $T$ ; two particles, each of mass  $m$ , are attached at the points of trisection, and a particle of mass  $M$  at the middle point. Shew that in small transverse oscillations one period is  $2\pi\sqrt{\left(\frac{2}{3}\frac{ml}{T}\right)}$ ; and that the other two periods cannot lie between this value and  $2\pi\sqrt{\left(\frac{Ml}{2T}\right)}$ .

**Question (1919 STEP III Q811)**

A light elastic spring of natural length  $l$  and modulus  $\lambda$  is lying just stretched on a smooth horizontal table. One end is attached to a point of the table and the other is attached to a particle of mass  $m$ . A blow of amount  $2\sqrt{m\lambda l/3}$  is applied to  $m$  perpendicularly to the spring. Find equations to determine the motion of  $m$ , and shew that in the subsequent motion the greatest length of the spring is  $2l$ .

**Question (1922 STEP III Q805)**

Prove that the small oscillations of a dynamical system about a position of equilibrium are compounded of a number of simple modes for each of which all coordinates of the system execute simple harmonic oscillations. Find the periods of oscillation in a vertical plane of a system consisting of two equal uniform rods  $AB, BC$  jointed together at  $B$  and hung from a joint at  $A$  so that they are vertical in their equilibrium position.