Question (1975 STEP I Q4)

k integers are selected from the integers 1, 2, ..., n. In how many ways is it possible if

(a) an integer once chosen may not be chosen again and regard is paid to the order of choice;

(b) an integer once chosen may not be chosen again but the order of choice is disregarded;

(c) the same integer may be chosen more than once and regard is paid to the order of choice;

(d) the same integer may be chosen more than once but the order of choice is to be disregarded?

Question (1982 STEP I Q11)

Seven sunbathers are positioned at equal intervals along a straight shoreline. Each stares fixedly at a nearest neighbour, choosing a neighbour at random if a choice is available. Show that the expected number of unobserved sunbathers is $\frac{3}{4}$.

Question (1980 STEP II Q6)

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with mean β, taking integer values in the range $1, 2, \ldots, K$. For each m, $1 \leq m \leq n$, let $S_m = X_1 + X_2 + \cdots + X_m$. Prove that $E(X_r/S_m) = 1/m$ for $r = 1, 2, \ldots, m$. Hence show that, if $m \leq n$, $E(S_m/S_n) = m/n$ and $E(S_n/S_m) = 1 + (n-m)\beta E(1/S_m)$.

Question (1981 STEP II Q6)

A and B play the following game. A throws two unbiased four-sided dice (each has the numbers 1 to 4 on its sides), and notes the total Y. B tries to guess this number, and guesses X. If B guesses correctly he wins X^2 pounds, and if he is wrong he loses $\frac{1}{2}X$ pounds. (a) Show that Bs average gain if he always guesses 8 is $\frac{1}{4}$. (b) He decides that he will always guess the same value of X. Which value of X would you advise him to choose, and what is his average gain in this case?

Question (1983 STEP II Q10)

A die is thrown until an even number appears. What is the expected value of the sum of all the scores?

Question (1968 STEP III Q3)

Let X be a random variable which takes on only a finite number of different possible values, say $x_1, x_2, \ldots, x_n$. Define the expectation of X, $E(X)$, and show that if a and b are constants then $E(aX+b) = aE(X)+b$. Define also the variance of X, $\text{var}(X)$, and similarly express $\text{var}(aX+b)$ in terms of $\text{var}(X)$. By considering separately those x_i which satisfy $|x_i - E(X)| > \epsilon$ and those which satisfy $|x_i - E(X)| \leq \epsilon$ where $\epsilon > 0$, show that

$$P[|X - E(X)| > \epsilon] \leq \frac{\text{var}(X)}{\epsilon^2}.$$

If $|x_i - E(X)| \leq \kappa$ for all i, where $\kappa > \epsilon$, show that

$$P[|X - E(X)| > \epsilon] \geq \frac{\text{var}(X) - \epsilon^2}{\kappa^2 - \epsilon^2}.$$

Question (1970 STEP III Q5)

Rain occurs on average on one day in ten. The weather forecast is 80% correct on days when it is really going to rain and 90% correct on days when it is going to be fine. On a particular day the forecast is rain. We have to decide whether to stay at home or whether to go out with or without an umbrella. The costs of these actions depend on whether or not it rains and are given in the following table.

	Rain	No rain
Stay at home	4	4
Go with umbrella	2	5
Go without umbrella	5	0

Which action is to be preferred?

Question (1979 STEP III Q9)

A computer prints out a list of M integers. Each integer has been chosen independently and at random from the range 1 to N, with equal probability assigned to each of the N possible values. What is the probability

(a) that the first r integers in the list are all different?

(b) that the rth integer is different from all the previous ones?

Show that the average number of different integers listed is $N\left[1 - \left(1 - \frac{1}{N}\right)^M\right]$.

Question (1976 STEP IIII Q10)

A standard pack of 52 cards is thoroughly shuffled, and then dealt into four piles as follows. Cards are dealt into the first pile up to and including the first ace, then into the second pile up to and including the second ace, then into the third pile up to and including the third ace, then into the fourth pile up to and including the fourth ace, and then any remaining cards go into the first pile again. A second similar pack is thoroughly shuffled, and a single card drawn from it at random. Find the probability distribution of the size of the pile that contains the matching card from the first pack.

Question (1959 STEP IIII Q111)

Craps is played between a gambler and a banker as follows. On each throw the gambler throws two dice. On the first throw he wins if the total is 7 or 11, but loses if it is 2, 3 or 12. If the first throw is none of these numbers, he subsequently wins if on some later throw he again scores the same as his first throw, but loses if he scores a 7. Calculate:

(i) the probability of winning on the first throw;

(ii) the probability of eventually winning with a 4;

(iii) the probability of winning.

Question (1966 STEP III Q11)

An investigator collects data on the expenditure in a given week of each of 300 households. He rounds off the figures to the nearest pound and takes the average. Assuming that for any one household the error he thus makes is equally likely to have any value between plus and minus 10 shillings, find the standard deviation of the departure of his answer from the true average.

Question (1922 STEP I Q112)

Shew that, if the base AB of a triangle ABC is fixed and the vertex C moves along the arc of a circle of which AB is a chord, then

$$\frac{da}{\cos A} + \frac{db}{\cos B} = 0.$$

Question (1915 STEP II Q305)

Prove that, when n is a positive integer,

$$\tan n\theta = \frac{n\tan\theta - \frac{1}{3!}n(n-1)(n-2)\tan^3\theta + \dots}{1 - \frac{1}{2!}n(n-1)\tan^2\theta + \frac{1}{4!}n(n-1)(n-2)(n-3)\tan^4\theta - \dots}.$$

Find an equation whose roots are the tangents of $\theta, 2\theta, 4\theta, 5\theta, 7\theta$ and 8θ where $\theta = 20°$, and shew that $\tan 20° \tan 40° \tan 80° = \sqrt{3}$.

Question (1934 STEP III Q502)

By means of the equation $(x+b)(x+c)-f^2=0$, prove that the equation in x

$$\begin{vmatrix} x+a & h & g \\ h & x+b & f \\ g & f & x+c \end{vmatrix} = 0$$

has three real roots which are separated by the two roots of the first equation. It may be assumed that a, b, c, f, g, h are all real and different from zero.