Question (1951 STEP II Q202)

If k and l are positive numbers, and the sequence (a_n) satisfies the recurrence relation

$$a_{n+1} = ka_n + la_{n-1},$$

prove that

$$\lim_{n\to\infty} \frac{a_n}{\alpha^n} = \frac{a_2 - \beta a_1}{\alpha(\alpha - \beta)},$$

where α is the positive root and β the negative root of the equation

$$x^2 - kx - l = 0.$$

Question (1944 STEP I Q104)

Find the sum to N terms of the series whose nth term is

$$\frac{1}{1+2+3+\cdots+n}.$$

Find the sum to infinity of the series whose nth term is

$$\frac{1+x+x^2+\cdots+x^{n-1}}{1+2+3+\cdots+n},$$

where x is numerically less than 1.

Question (1948 STEP I Q101)

If $u_0 = 1, u_1 = 2$ and

$$u_{n+2} = 2(u_{n+1} - u_n) \quad (n = 0, 1, 2, \dots),$$

show that $u_{4k} = (-4)^k$ and find u_{4k+1}, u_{4k+2} and u_{4k+3}. Prove that

$$\sum_{n=1}^{4k} u_n^2 = \frac{2}{3}(16^k - 1).$$

Question (1948 STEP II Q403)

A recurring series whose nth term is u_n has the scale of relation:

$$u_{n+3} - 6u_{n+2} + 11u_{n+1} - 6u_n = 0.$$

Show that u_n is of the general form

$$3^n A + 2^n B + C,$$

where A, B, C are independent of n. Find the value of the nth term if $u_1 = 1, u_2 = 6, u_3 = 14$.

Question (1933 STEP I Q104)

Sum the series, n being a positive integer:

(i) $$\frac{1}{(2n)!^2} + \frac{1}{(2n-2)!(2n+2)!} + \frac{1}{(2n-4)!(2n+4)!} + \cdots + \frac{1}{2!(4n-2)!} + \frac{1}{(4n)!},$$

(ii) $1 + x\cos\theta + x^2\cos 2\theta + \cdots + x^n \cos n\theta$.

Question (1913 STEP I Q103)

Shew that the square of any even number $2n$ is equal to the sum of n terms of a series of integers in Arithmetical Progression; and that the square of any odd number $2n+1$ exceeds by unity the sum of n terms of another such progression.

Question (1940 STEP I Q105)

(i) If x is positive and not equal to 1 and p is rational and not equal to 0 or 1, prove that $x^p - 1$ is less than or greater than $p(x-1)$ according as p is between 0 and 1 or is outside these limits.

(ii) If $a_1, a_2, \ldots, a_n$ are positive, show that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{1/n}.$$

Prove that, if x, y, z are positive and $x + y + z = 1$, the greatest value of $x^2y^7z^6$ is $2^{10}/3^{15}$.

Question (1917 STEP II Q203)

Prove, by means of the identity $\frac{p}{1-px} - \frac{q}{1-qx} = \frac{p-q}{(1-px)(1-qx)}$, or otherwise, that, if n be an even integer,

$$(-1)^{\frac{1}{2}n}\frac{p^{n+1} - q^{n+1}}{p-q} = (pq)^{\frac{1}{2}n} - \frac{(n+1)^2 - 1^2}{2.4}(pq)^{\frac{1}{2}n-1}(p+q)^2 + \ldots$$
$$+ (-1)^r \frac{\{(n+1)^2 - 1^2\}\{(n+1)^2 - 3^2\}\ldots\{(n+1)^2 - (2r-1)^2\}}{2.4.6\ldots 4r}(pq)^{\frac{1}{2}n-r}($$

Question (1929 STEP II Q204)

Prove that

$$\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\cdots+\sin(\alpha+n\beta)=\frac{\sin(\alpha+\frac{n+1}{2}\beta)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

and deduce the sum of

$$\sin\theta-\sin 2\theta+\sin 3\theta-\cdots-\sin 2r\theta.$$

Shew also that

$$\cos^2 x+\cos^2 2x+\cdots+\cos^2 nx=\frac{2n-1}{4}+\frac{\sin(2n+1)x}{4\sin x}.$$

Question (1934 STEP I Q304)

(i) Sum to n terms the series

$$\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\dots.$$

(ii) Prove that

$$\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\dots \text{ to infinity}=5e-1.$$

Question (1922 STEP II Q403)

Find the sum of the squares and cubes of the first n odd integers. Shew that the sum of the products two at a time of the first n odd integers is

$$\tfrac{1}{3}n(n-1)(3n^2-n-1).$$

Question (1916 STEP II Q502)

Express $\tan n\theta$ in terms of $\tan\theta$ when n is a positive integer. Prove that

$$\sum_{r=1}^{r=10}\operatorname{cosec}^2\frac{r\pi}{11}=40.$$

Question (1922 STEP II Q502)

Along a straight line are placed n points. The distance between the first two points is one inch; and the distance between the rth and $(r+1)$th points exceeds one inch by $\frac{1}{p}$th of the distance between the $(r-1)$th and rth points, for all values of r from 2 to $(n-1)$. Find the distance between the first and last points. Shew that, if n is large, the distance approximates to $\frac{pn}{p-1}-\frac{p}{(p-1)^2}$ inches; and shew that, if $n=p=10$, the distance is exactly 9.87654321 inches.

Question (1926 STEP II Q501)

Find the sum of the cubes of the first n integers, and show that if m is the arithmetic mean of any n consecutive integers, the sum of their cubes is

$$mn\{m^2 + \tfrac{1}{4}(n^2 - 1)\}.$$

Prove that if s_1, s_2, s_3 are the sums of the first, second and third powers of any consecutive integers $9s_2^2 > 8s_1s_3$.

Question (1913 STEP II Q702)

Prove that the geometric mean between two quantities is also the geometric mean between their arithmetic and harmonic means. Sum the series

$$a + (a+b)r + (a+2b)r^2 + (a+3b)r^3 + \cdots + (a + \overline{n-1}b)r^{n-1}.$$