Question (1982 STEP I Q16)

Particles of mud are thrown off the tyres of the wheels of a cart travelling at constant speed V. Neglecting air resistance, show that a particle which leaves the ascending part of a tyre at a point above the hub will be thrown clear of the wheel provided its height above the hub at the instant when it leaves the tyre is greater than a^2/V^2, where a is the radius of the tyre.

None

Question (1976 STEP IIII Q13)

A particle is placed inside a fixed smooth hollow sphere of internal radius a. It is projected horizontally from the lowest point with speed u. Show that it will leave the surface in the subsequent motion provided that

$$2ag < u^2 < 5ag.$$

Show that the time that elapses between the particle leaving the surface and subsequently striking it again is greatest when

$$u^2 = ag(2 + \sqrt{3}).$$

None

Question (1964 STEP IIII Q110)

A light rod OA of length l rotates freely about a fixed point O. A point particle of mass m attached to the rod at A is initially at rest vertically below O. A projectile of mass m moving horizontally with speed v ($v^2 < 16gl$) embeds itself instantaneously in the target. Obtain the height h through which the target would rise before first coming to rest if undisturbed in the subsequent motion. However, after rising through a height $3h/4$ another similar projectile embeds itself in the target. How much further will the target rise? If the total height through which the target rises is $3h/4 + h'$, show that h' is greatest (for variable v) if $v^2 = 16gl/3$.

None

Question (1962 STEP III Q103)

A particle is released from rest at a point on the surface of smooth sphere very near to the top. Find where it leaves the sphere. If the sphere is roughened in patches and the particle, released in the same way, eventually leaves the surface, prove that it does so at a lower point than in the previous case.

None

Question (1958 STEP III Q308)

A bead of mass m slides on a smooth wire in the form of a circle of radius a which is fixed in a vertical plane. The bead is projected from the lowest point of the circle at the instant $t = 0$ with velocity $2\sqrt{(ga)}$, and in the subsequent motion the radius from the centre of the circle to the bead makes an angle θ with the downward vertical at time t. Prove that

$$\sin\frac{\theta}{2} = \tanh nt,$$

where $n^2 = g/a$. If R is the reaction of the wire on the bead at any time during the motion, R being measured towards the centre of the circle, express R (i) as a function of θ, and (ii) as a function of t.

None

Question (1961 STEP III Q305)

A particle falls from a position of limiting equilibrium near the top of a nearly smooth glass sphere. Show that it will leave the sphere at a point whose radius is inclined to the vertical at an angle

$$\alpha + \mu\left(2 - \frac{4}{3\sin\alpha}\right), \tag{1}$$

approximately, where $\cos\alpha = \frac{2}{3}$ and μ is the coefficient of friction.

None

Question (1959 STEP III Q404)

A particle is attached to one end of a light perfectly flexible string of length a whose other end O is fixed. When hanging at rest the particle is given a horizontal velocity u. Find conditions to ensure that O will be the lowest point at which, in the subsequent motion the string remains taut, and show that if these conditions are not satisfied the particle will pass through O if $u^2 = (2 + \sqrt{3})ga$.

None

Question (1966 STEP III Q8)

A wire in the form of a circle of diameter $6a$ is fixed in a vertical plane. A bead of mass m is connected to the highest point by means of an elastic string of natural length $3a$ which exerts a force $\lambda(l - 3a)$ when stretched to length l, where $\lambda = 2mg/a$. The bead is initially sliding down the wire, and when its angular distance 2θ from the lowest point is $120°$, so that the string becomes taut, its speed is $3\sqrt{(ga)}$. Show that it will continue moving down till it reaches the bottom and that its speed will then be $4\sqrt{(ga)}$. Find also how long it takes to get there.

None

Question (1957 STEP III Q108)

One end A of a light elastic string of natural length a and modulus of elasticity λ is fixed. The other end is attached to a particle of mass m which moves on a smooth horizontal table at a depth b below A. If the particle moves in a circle with constant angular velocity ω and with the string inclined at a constant angle α to the vertical, prove that

$$b\omega^2 \leq g, \quad mab\omega^2 = \lambda(b - a\cos\alpha).$$

Deduce that ω must satisfy the conditions

$$\lambda(b-a) < mab\omega^2 < \lambda b$$

and that no such motion (whatever the values of ω and α) is possible if the particle can hang in equilibrium without reaching the table.

Question (1957 STEP III Q207)

A simple pendulum is making complete revolutions in a vertical plane in such a way that its greatest and least angular velocities are ω_1 and ω_2 respectively. Show that when the inclination of the pendulum to the downward vertical is θ the angular velocity is

$$(\omega_1^2 \cos^2 \tfrac{1}{2}\theta + \omega_2^2 \sin^2 \tfrac{1}{2}\theta)^{\frac{1}{2}}.$$

Show further that stationary values of the tension can never occur except at the highest and lowest positions, and find the corresponding formula for the tension at a general position in terms of its greatest and least values T_1 and T_2.

Question (1957 STEP III Q310)

A uniform rod of length l and mass m swings in a plane under gravity about one end where it is freely hinged. Given that the maximum deflection from the vertical is α, obtain (a) the angular velocity, and (b) the horizontal and vertical reactions at the hinge, when the rod makes an angle θ with the vertical.

Question (1944 STEP IIII Q109)

A uniform heavy chain of length 10 feet is given two complete turns and a half turn round a smooth circular cylinder of diameter 1 foot whose axis is horizontal. The chain is to be assumed to lie in a vertical plane perpendicular to the axis of the cylinder and its free ends to hang symmetrically one on each side of the cylinder. Investigate whether the chain remains in contact with the lowest generator of the cylinder.

Question (1944 STEP III Q110)

A flywheel with radius r and moment of inertia I is mounted in smooth bearings with its axle horizontal. The flywheel being at rest, an inelastic particle of mass m, falling vertically with velocity v, strikes the rim at a point where the radius makes an angle α with the vertical, and adheres without rebound. Determine the angular velocity of the flywheel immediately after the impact, and also when it has turned through an angle $\frac{3}{2}\pi - \alpha$.

Question (1945 STEP III Q106)

A particle rests on top of a smooth fixed sphere. If the particle is slightly displaced, find where it leaves the surface. Find also where it crosses the horizontal plane through the centre of the sphere.

Question (1944 STEP III Q210)

Two particles of masses $4m, 3m$ connected by a taut light string of length $\frac{1}{2}\pi a$ rest in equilibrium on a smooth horizontal cylinder of radius a. If equilibrium is slightly disturbed so that the heavier particle begins to descend, find at what point it will leave the surface, and shew that at that moment the pressure on the other particle is slightly greater than two-thirds of its weight. [Trigonometrical tables should be used.]

Question (1946 STEP III Q308)

A simple pendulum consists of a particle of mass m attached to a fixed point O by a light inelastic string of length a. The particle moves in a complete vertical circle in such a way that the tension in the string just vanishes at the highest point. What is the tension at the lowest point? Prove that the greatest value of the horizontal component of the tension during the motion is $9\sqrt{3}mg/4$.

Question (1944 STEP III Q406)

A smooth wire is bent into the form of a circle of radius a and is held with its plane inclined to the horizontal at an angle α. A small bead, projected with speed $(\frac{8}{3}ga\sin\alpha)^{\frac{1}{2}}$ from the lowest point of the wire, moves on the wire. Prove that if $\alpha > \frac{\pi}{6}$ the resultant reaction between the bead and the wire is horizontal in two and only two positions of the bend.

Question (1945 STEP III Q408)

A particle can move freely on a horizontal table inside a circular barrier of radius a formed by a circular cylinder fixed to the table with its axis vertical. The particle is projected with velocity V along the table from a point A of the barrier along a chord AB subtending an angle 2α $(< \pi)$ at the centre. The coefficient of restitution between the particle and the barrier is e. Show that the particle ultimately reaches a steady state of motion circulating uniformly round the inside of the barrier in a time less than $\frac{2a\sin\alpha}{V\cos^2\alpha}\frac{1}{1-e}$. What is the velocity of this circular motion? Discuss briefly the case $\alpha = \pi/2$.

Question (1946 STEP III Q405)

A heavy particle is attached by two light strings of lengths a and b to two points in the same vertical line at distance c apart such that $c^2 < a^2 - b^2$. If the particle describes a horizontal circle with constant angular velocity ω, show that both strings will be taut provided

$$\frac{\omega^2}{2gc}(a^2 - b^2 + c^2)^{-1} < 1 < \frac{\omega^2}{2gc}(a^2 - b^2 - c^2)^{-1}.$$

What are the corresponding conditions if $c^2 > a^2 - b^2$?

Question (1946 STEP III Q406)

State Newton's law relating to impact between imperfectly elastic bodies. A circular hoop of mass M is free to swing in a vertical plane about a frictionless horizontal pivot passing through a point O of its circumference. The hoop is hanging in equilibrium when a smooth spherical ball of mass m falls vertically and strikes it at a point P at angular distance θ (acute) from O. If the ball rebounds horizontally in the vertical plane of the hoop, show that the coefficient of restitution, e, between the ball and the hoop is

$$\left(1 + \frac{m}{2M}\right)\tan^2\theta.$$

What would be the requisite value of e for horizontal rebound to occur if the hoop were made immovable?

Question (1923 STEP I Q107)

An equilateral triangle ABC is drawn on an inclined plane. The heights of A, B, C above a horizontal plane are as $1 : 12 : 14$. Show that the side BC makes an angle $\sin^{-1}(1/7)$ with a horizontal line drawn on the inclined plane.

Question (1914 STEP I Q112)

Two small rings of masses m, m' are moving on a smooth circular wire which is fixed with its plane vertical. They are connected by a straight massless inextensible string. Prove that, while the string remains tight, its tension is $2mm'g\tan\alpha\cos\theta/(m+m')$, where 2α is the angle subtended by the string at the centre of the ring, and θ is the inclination of the string to the horizon.

Question (1914 STEP I Q114)

A heavy particle slides down a smooth vertical circle of radius R from rest at the highest point. Shew that on leaving the circle it moves in a parabola whose latus rectum is $\frac{16}{27}R$.

Question (1920 STEP I Q107)

A disc is rotated about its axis, which is vertical, from rest with uniform angular acceleration α. A particle rests on it at distance a from the centre of the disc. The coefficient of friction between disc and particle is μ. After a time the particle slips; when does this happen and in what direction over the disc does slipping begin?

Question (1913 STEP I Q110)

A heavy particle hangs by a string of length a from a fixed point O and is projected horizontally with the velocity due to falling freely under gravity through a distance h; prove that if the particle makes complete revolutions $h \geq \frac{5}{2}a$, that if the string becomes slack $\frac{5}{2}a > h > a$, and that in this latter case the greatest height reached above the point of projection is $(4a-h)(a+2h)^2/27a^2$.

Question (1914 STEP I Q117)

Show that the surface generated by the revolution of the cardioid

$$r = a(1-\cos\theta)$$

about the line $\theta = 0$ is $\dfrac{32}{5}\pi a^2$.

Question (1916 STEP I Q115)

A particle of mass m is tied to the middle point of a light string 26 inches long, whose ends are attached to fixed points in the same horizontal line, 2 ft. apart. If the particle rotates about this line so that it describes a complete circle in a vertical plane, shew that the tension of the string, when the particle is at its lowest point, is necessarily more than $(7.8)mg$.

Question (1917 STEP I Q114)

A flywheel weighing 40 lbs. has a radius of gyration 9 inches; it is driven by a couple fluctuating during each revolution so that the curve connecting couple and angular position during one revolution is a triangle, the couple becoming zero once per revolution and reaching a maximum of 2 ft. lbs. There is also a constant resisting couple such that the motion is the same in each revolution. If the flywheel makes 60 revolutions a minute, shew that the difference between the greatest and least angular velocities is approximately 5.7 per cent. of either.

Question (1926 STEP I Q104)

A solid homogeneous circular cylinder of radius r is bisected by a plane passing through its axis and on one half as base is constructed a triangular prism of isosceles section and of the same substance: the whole is placed in equilibrium on the top of a fixed circular cylinder of radius $2r$ with axis horizontal – the axes of the cylinders being parallel and the curved surfaces in contact. Shew that the greatest height of the prism consistent with stability for a small rolling displacement is

$$r\frac{\sqrt{9-2\pi}-1}{2}.$$

[N.B. The centre of gravity of a semi-circle of radius r is distant $4r/3\pi$ from the centre of the semi-circle.]

Question (1927 STEP I Q112)

State the principle of the conservation of angular momentum of a system about a fixed axis. A flywheel of moment of inertia I is rotating with angular velocity Ω about a vertical axis. The flywheel contains a pocket at a distance a from the axis into which is dropped a sphere of mass M, moment of inertia i and spin ω about a vertical axis, without horizontal motion. Find the angular velocity of the system after the sphere has come to relative rest in the pocket.

Question (1928 STEP I Q106)

A horizontal portion of a toboggan run is worn into a series of sine-curve undulations 20 ft. from crest to crest, with a maximum height from crest to trough of 1 in. Shew that a toboggan will jump at the crest of an undulation when its speed exceeds about 60 miles an hour.

Question (1928 STEP I Q108)

A heavy particle is attached to the rim of a wheel of radius r which is made to rotate in a vertical plane with constant angular velocity ω about its centre which is fixed. Shew that if the particle is set free at some point of its path the time t taken to reach the horizontal plane through the lowest point of the wheel is given by $g^2t^4 - 2t^2r(g+r\omega^2) + x^2 = 0$, where x is the horizontal distance traversed measured from the lowest point of the wheel. Deduce that the greatest value of x is $r + \omega^2r^2/g$.

Question (1931 STEP I Q109)

Explain how to reduce the solution of a dynamical problem to that of a statical problem. A uniform rod of length l attached at one end to a fixed point by a smooth universal joint rotates freely under gravity as a conical pendulum. If ω is the angular velocity of the vertical plane through the rod, and α the constant inclination of the rod to the vertical, prove that

$$\omega^2 = \frac{3g}{2l}\sec\alpha.$$

Question (1936 STEP I Q107)

A particle is free to move on a smooth vertical circle of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Shew that, after a time

$$\sqrt{\frac{a}{g}}\log_e(\sqrt{5}+\sqrt{6}),$$

the reaction between the particle and the wire is zero.

Question (1942 STEP I Q106)

An elastic ring of mass M, natural length $2\pi a$, and modulus of elasticity λ is placed upon a rough, horizontal, steadily rotating turntable; the coefficient of friction is μ. Find the range of values of ω, the angular velocity of the turntable, for which the ring can remain on the turntable, without slipping, in the form of a circle of radius $2a$ with its centre at the centre of rotation.

Question (1920 STEP I Q110)

A particle moves under gravity on a given smooth curve in a vertical plane; shew how to determine the velocity and the pressure on the particle at any point. The particle describes a complete circle and the pressure is always directed inwards; shew that, if the maximum pressure is to the minimum pressure as m to n, the maximum velocity is to the minimum velocity as $(5m+n)^{\frac{1}{2}}$ to $(m+5n)^{\frac{1}{2}}$.

Question (1922 STEP I Q109)

Prove the principles of Conservation of Momentum and of Kinetic Energy for a material system. A particle of mass m is placed at the vertex of a hemisphere of mass M, whose base rests on a smooth horizontal plane, and is slightly disturbed. Shew that the path in space of the particle is an arc of an ellipse, as long as contact is preserved. Shew that contact ceases when the radius to the particle makes an angle θ with the vertical given by the equation

$$m\cos^3\theta - 3(M+m)\cos\theta + 2(M+m) = 0;$$

and verify that if $M = m\tan^2\alpha$ ($\alpha < \pi/2$), the value to be taken for $\cos\theta$ is

$$2\sec\alpha\cos\frac{\pi+\alpha}{3}.$$

Question (1926 STEP I Q108)

Define the angular velocity of a lamina moving in any manner in its plane and shew how to determine it when the velocities of two points of the lamina are given. A circle A of radius a turns round its centre with angular velocity ω. A circle B of radius b rolls on the circle A and its angular velocity is ω'. Find the time taken

(1) for the point of contact to make a complete circuit of A,

(2) for the centre of B to return to a former position.

Determine the accelerations of the common point of the two circles and the greatest acceleration of a point on the circle B.

Question (1929 STEP I Q109)

Two particles of equal mass joined by a light inextensible string of length $\pi r/2$ rest in (unstable) equilibrium on the outer surface of a smooth circular cylinder of radius r whose axis is horizontal. If the equilibrium is slightly disturbed and the particles begin to move in a vertical plane, prove that the first particle to leave the surface of the cylinder does so when the perpendicular drawn from it to the axis of the cylinder is inclined at about 13° to the horizontal.

Question (1929 STEP I Q110)

A uniform circular wire of mass m and radius r can rotate freely about a fixed vertical diameter, and a small ring of mass m can move freely along the wire. The wire is started rotating with angular velocity ω, at an instant when the ring is at one end of a horizontal diameter and is at rest with respect to the wire: if the system is left to itself in the subsequent motion, shew that, provided $\omega^2 < 2g/3r$, the ring reaches the lowest point of the wire with velocity

$$(2gr - 3\omega^2 r^2)^{\frac{1}{2}}.$$

Describe the motion of the ring if $\omega^2 > 2g/3r$.

Question (1938 STEP I Q109)

A number of equal masses m are joined by light strings of length s so that the masses are at the angular points of a regular polygon, of side s, inscribable in a circle of radius a. The polygon lies on a smooth horizontal table and rotates steadily in its own plane with angular velocity ω about its centre. Show that the tensions in the strings are $ma^2\omega^2/s$ in absolute units. By considering the limiting case in which the number of sides tends to infinity, find the tension in a uniform string, of mass $2\pi a\lambda$, which is in the form of a horizontal circle of radius a and is rotating in its own plane about its centre with uniform angular velocity ω. Verify this latter result by introducing centrifugal forces and applying the principle of virtual work to the corresponding statical problem.

Question (1916 STEP I Q110)

State and prove the acceleration property of the hodograph. Determine the hodographs of (1) a projectile describing a parabolic path, (2) a particle on the inside of the rim of a cartwheel which is travelling with uniform speed along a straight road. The thickness of the rim is to be taken as $\frac{1}{10}$ of its external radius. In each case draw also the actual paths of the bodies, and indicate corresponding points on path and hodograph. A radial force of $1\frac{1}{2}$ ounces weight is needed to detach the particle, weighing 2 oz., from the rim. If the cart is travelling at 4 miles an hour, and the particle is in limiting equilibrium at the top of its path, find the radius of the wheel.

Question (1914 STEP I Q210)

A particle describes a circle with variable speed. Find the tangential and normal components of the force on the particle. AB is the upper side (a) of the square cross-section of a log which has two sides of the section vertical. A particle is attached to A by a string, of length $l(> 4a)$, which is initially stretched out along BA produced. Prove that, if the particle is projected downwards with velocity greater than $\{g(3l - 8a)\}^{\frac{1}{2}}$, the string will wrap tightly round the log till the particle strikes the log.

Question (1917 STEP I Q210)

Find the acceleration of a point describing a circle with variable velocity. Two beads connected by a string are held at rest on a vertical circular wire with the string horizontal, and above the centre. Their masses are m, m', and the string subtends an angle 2α at the centre. If the beads are released, show that the tension of the string when it makes an angle θ with the horizontal is

$$\frac{2mm'g\tan\alpha\cos\theta}{m+m'}.$$

Question (1920 STEP I Q209)

A particle tied to a fixed point O by an inextensible string of length a is projected horizontally from the lowest position so that the string becomes slack when the particle is at height h above O. Find the velocity of projection, and prove that the string will again become taut after an additional time

$$4\sqrt{\frac{h}{g}\left(1-\frac{h^2}{a^2}\right)}.$$

Question (1923 STEP I Q210)

Show that any possible motion of a system of particles still satisfies the equations of motion if the same uniform velocity is compounded with the velocity of every particle and the forces between the particles remain unaltered. A pendulum consisting of a light rod of length l and a heavy bob hangs freely. The point of support is suddenly made to move horizontally with uniform velocity v. Show that the pendulum will describe a complete revolution if $v > 2\sqrt{(gl)}$.

Question (1927 STEP I Q204)

Shew that for a lamina moving in a plane there is in general an instantaneous centre of zero velocity. Shew further that for two laminas moving in the same plane there is in general a relative instantaneous centre having the same velocity for both, and that it lies in the join of the instantaneous centres of the two laminas. A link C_1C_2 is hinged to points C_1, C_2 on rods A_1B_1, A_2B_2 respectively. A_1, A_2 move on a straight line Ox, and B_1, B_2 on a perpendicular straight line Oy. Find the relative instantaneous centre of the rods.

Question (1927 STEP I Q209)

Find the radial and transverse components of acceleration of a point moving in a circle. A smooth solid circular cylinder of radius a is fixed in contact along its lowest generator with a horizontal plane. A particle slides on the cylinder in a plane perpendicular to the generators. If the particle leaves the cylinder at an angular distance θ from the highest generator, prove that it meets the plane at a distance x from the line of contact given by

$$\frac{x}{a} = \sin^3\theta + \cos\theta(1+\cos\theta)\sqrt{\cos\theta(2-\cos\theta)}.$$

Question (1928 STEP I Q210)

OA, AB are two inextensible strings each of length 5 ft. O is attached to a fixed point and masses m_1, m_2 are attached at A and B. The whole swings round so that both strings lie in a vertical plane which rotates about the vertical through O with uniform angular velocity ω. Shew that if A and B remain at distances 3 ft. and 7 ft. respectively from the axis of rotation the ratio m_1/m_2 is equal to 49/15, and find the necessary value of ω.

Question (1930 STEP I Q209)

A heavy particle of weight W, attached to a fixed point by a light inextensible string, describes a circle in a vertical plane. The velocities at the highest and lowest points are V and nV respectively. Shew that, when θ is the inclination of the string to the downward vertical, the tension in the string is

$$\left\{\frac{2(n^2+1)}{n^2-1} + 3\cos\theta\right\} W.$$

Hence find the limits within which n must lie if the string cannot support a tension greater than $9W$.

Question (1931 STEP I Q206)

A heavy particle is attached to a fixed point O by a light elastic string of natural length l. When the string is vertical and the system at rest, the length of the string is l/ν. If the particle describes a horizontal circle about the vertical through O with uniform angular velocity $(g\kappa/l)^{\frac{1}{2}}$, prove that the radius of the circle is

$$l\{v^2\kappa^2 - (\nu + \nu\kappa - \kappa)^2\}^{\frac{1}{2}}/\kappa(\nu + \nu\kappa - \kappa),$$

and that κ must lie between ν and $\nu/(1-\nu)$.

Question (1932 STEP I Q208)

A particle hangs by an inelastic string of length a from a fixed point, and a second particle of the same mass hangs from the first by an equal string. The whole moves with uniform angular velocity ω about the vertical through the point of suspension, the strings making constant angles α and β with the vertical. Shew that

$$\tan\alpha = \frac{a\omega^2}{g}(\sin\alpha + \frac{1}{2}\sin\beta),$$

and

$$\tan\beta = \frac{a\omega^2}{g}(\sin\alpha + \sin\beta).$$

Hence shew that if α and β are small such a steady motion is only possible if $a\omega^2$ has one of the values

$$(2 \pm \sqrt{2})g$$

and that $\beta/\alpha = \pm\sqrt{2}$.

Question (1934 STEP I Q209)

On a smooth plane inclined at an angle α to the horizontal a particle is lying at rest attached to a fixed point above the plane by an inextensible string making an acute angle β with the plane. Prove that it is possible to project the particle so that it describes a complete circle on the plane if $\cot\alpha \geq 6\tan\beta$.

Question (1935 STEP I Q209)

Find the radial and transverse accelerations of a particle moving in a plane, referred to polar coordinates. A particle of unit mass is attached to one end of an elastic string of natural length l and modulus of elasticity λ whose other end is fixed. The particle is held so that the string is just tight, and projected at right angles to the string with velocity $2\sqrt{\frac{\lambda l}{3}}$. Show that the greatest length of the string in the resulting motion is $2l$.

Question (1938 STEP I Q206)

A particle slides from rest at the vertex of a smooth surface formed by revolving a parabola about its axis, the axis being vertical and the vertex upwards. Prove that the particle remains in contact with the surface. If v and f are the horizontal components of the velocity and acceleration respectively at any point, shew that the reaction at the point is proportional to f/v.

Question (1930 STEP II Q207)

Assuming that $\pi[ab - h^2]^{-\frac{1}{2}}$ is the area of the ellipse $ax^2 + 2hxy + by^2 = 1$, shew that the minimum area of an ellipse which passes through the points $(\pm p, 0)$, (q, r) and $(-q, -r)$ is equal to πpr.

Question (1933 STEP III Q206)

PQ is a focal chord of a parabola and the normals at P and Q meet the parabola again at P' and Q'. Shew that PQ and $P'Q'$ are parallel and that the ratio of their lengths is $1 : 3$.

Question (1919 STEP IIII Q208)

Show that, if a point moves along any curve under the action of a force always at right angles to the direction of motion, the point moves with constant speed. A particle is attached to the end of a string which is partly wound round a post whose section is a regular polygon of r sides each of length a. Initially a length l, equal to an integral multiple of a, is unwound and in a straight line with one of the sides. The particle is then projected at right angles to the string with velocity v so that the string winds in a horizontal plane round the post. Show that the time taken to wind up is

$$\frac{\pi l(l + a)}{rav}.$$

Question (1921 STEP IIII Q207)

A convex quadrilateral is inscribed in a circle of given radius R, and one side subtends a given angle α at a point of the arc of the circle on the opposite side to the quadrilateral. Prove that the greatest area of the quadrilateral is

$$2R^2 \sin^3 \frac{2}{3}\alpha.$$

Question (1930 STEP IIII Q208)

A bead slides under gravity along a smooth straight wire. Shew that if the bead starts from rest at a given origin P the locus of positions after a given time for different directions of the wire in a vertical plane through P is a circle. Find the straight line of quickest descent to a circle in a vertical plane from a point P of the plane. Shew also that if the time of quickest descent from P to the circle is prescribed the locus of P is a circle. Give a geometrical construction for the straight line of quickest descent from one circle to another, external to it, in the same vertical plane.

Question (1936 STEP IIII Q208)

A particle is projected along the outside surface of a smooth sphere of radius a ft. from the highest point with velocity $\frac{1}{2}\sqrt{(ga)}$. Prove that it strikes a horizontal plane through the centre of the sphere at a distance

$$\frac{9\sqrt{39}+7\sqrt{7}}{64}a \text{ ft.}$$

from the centre.

Question (1920 STEP II Q309)

Prove that the acceleration towards the centre of a particle moving in a circle is v^2/r. Two particles describe two circles in a plane uniformly in the same time. Prove that the acceleration of one relative to the other is constant in magnitude and changes its direction uniformly.

Question (1920 STEP III Q314)

A uniform hemisphere of given mass rests on a smooth horizontal plane and a smooth perfectly elastic particle of mass equal to that of the hemisphere is dropped vertically so as to strike the hemisphere with a given velocity. Shew that in order that the velocity of the hemisphere after impact may be a maximum, the point of impact must be at an angular distance $\sin^{-1}(1/\sqrt{3})$ from the highest point of the hemisphere.

Question (1921 STEP III Q314)

A smooth wire is bent into the form $y = \sin x$ and placed in a vertical plane with the axis of x horizontal. A bead of mass m slides down the wire starting from rest at $x = \frac{\pi}{2}$. Shew that the pressure on the wire as the bead passes through the origin is $mg/\sqrt{2}$, and find the pressure as it passes through $x = -\frac{\pi}{2}$.

Question (1926 STEP III Q308)

If a particle is describing a circle of radius r with uniform speed v, prove that the acceleration is $\frac{v^2}{r}$ towards the centre. A particle hanging by a light string of length l from a fixed point O is projected horizontally from its lowest position with velocity $\sqrt{\frac{7}{2}gl}$. Prove that the string slackens after swinging through $120°$.

Question (1927 STEP III Q308)

A is the highest point of a fixed smooth sphere whose centre is O. A particle P, starting from rest at A, slides under the action of gravity down the outside of the sphere. Prove that it will leave the sphere when $3\cos\theta = 2$, where θ is the angle AOP. For any smaller value of θ, find the radial and tangential components of the acceleration of the particle.

Question (1931 STEP III Q306)

In a smooth fixed circular tube, of radius a and small bore, in a vertical plane, are two particles of masses m and $2m$, connected by a light inextensible string of length $a\pi$. With the particles at the ends of the horizontal diameter, and the string in the upper half of the tube, the system is released from rest. Prove that, when each particle has described an arc $a\theta$ $\left[\theta < \frac{\pi}{2}\right]$, the pressure between the lighter particle m and the tube is $\frac{1}{3}mg\sin\theta$, and the tension of the string is $\frac{2}{3}mg\cos\theta$.

Question (1932 STEP III Q306)

A particle slides down the outside of a fixed smooth sphere of radius r, starting from rest at a height $\frac{1}{8}r$, measured vertically, above the centre. Prove that it leaves the sphere when at a height $\frac{1}{4}r$ above the centre. Prove also that when the particle is at a horizontal distance $r\sqrt{2}$ from the centre, it is at a vertical depth $4r$ below the centre.

Question (1937 STEP III Q307)

A small spherical ball B, of mass m, hangs at rest under gravity at the end of a light inextensible string AB of length a which is fixed to a rigid support at A. A second spherical ball of mass M impinges on the first with velocity V, the velocity and the line of centres of the two spheres both being horizontal at the instant of impact. The string AB can support a tension of seven times the weight of B, and the coefficient of restitution between the two balls is e. Shew that, after the impact, B describes complete circles about A provided that $V_0 < V \leq V_1$, where V_0 and V_1 are certain fixed velocities. Determine V_0 and V_1 and explain what happens in the two cases $V < V_0$ and $V > V_1$.

Question (1941 STEP III Q310)

A uniform spherical ball of radius a is at rest on a rough horizontal table, and is set in motion by a horizontal blow in a vertical plane through the centre at a distance $\frac{2}{3}a$ above the table. Show that when the ball ceases to slip its linear velocity is $5/14$ of its initial linear velocity.

Question (1938 STEP IIII Q308)

A particle is attached by a light inextensible string of length a to a fixed point. The particle hangs in equilibrium and is then given a horizontal velocity $\sqrt{(7ga/2)}$. Show that during the subsequent motion the maximum height of the particle above its initial position is $27a/16$.

Question (1939 STEP IIII Q310)

Show that when a particle describes a curve its acceleration components along and perpendicular to the curve are $\frac{dv}{dt}$ and $\frac{v^2}{\rho}$, where v is the velocity of the particle, and ρ is the radius of curvature at the instantaneous position of the particle.

Two equal particles are connected by a light inelastic string of length πa, and are placed on a smooth circular cylinder of radius $2a$ which has its axis horizontal, so that they rest in unstable equilibrium with the string passing over the top of the cylinder. If the equilibrium is slightly disturbed, find the tension in the string and the reactions of the particles on the cylinder in terms of the angular displacement, and show that the lower particle leaves the cylinder when at an angular distance of approximately 77° 20' from the top.

Question (1940 STEP IIII Q308)

A particle of mass m is constrained to move on a smooth wire in the shape of a parabola whose axis is vertical and whose vertex is upwards. The particle is projected from the vertex with velocity u. Show that the pressure on the wire at any point is

$$\frac{m}{\rho}(gl - u^2),$$

where $2l$ is the latus rectum, and ρ is the radius of curvature.

Question (1942 STEP IIII Q308)

A particle hangs from a light inextensible string of length r attached at its upper end to a point on a vertical wall. It is projected with velocity $2\sqrt{rg}$ perpendicular to the wall. Find the point at which the particle will subsequently hit the wall.

Question (1926 STEP I Q408)

The centre of a fixed circle of radius $\frac{3}{2}r$ is on the circumference of another fixed circle of radius r. Inside the smaller crescent-shaped area intercepted between the circles is placed a movable circle of radius $\frac{1}{2}r$. If this circle remains in contact with the circle of radius r, prove that the length of arc described by its centre in moving from one extreme position to the other is $\frac{13}{12}\pi r$.

Question (1937 STEP I Q410)

A cylindrical body of any section can turn freely about a fixed horizontal axis which is parallel to its generators. While at rest under gravity the body is subjected to a certain horizontal blow perpendicular to the axis and at a point beneath such that in the subsequent motion the kinetic energy of the body is proportional to the depth of its centre of gravity below its highest possible position. With the usual notation (M, h, κ^2) find the time taken from rest until the centre of gravity is on the same horizontal level as the axis. If in addition to these circumstances it is further observed that there is no impulsive reaction at the axis of suspension at impact, determine the magnitude of the blow given.

Question (1939 STEP I Q408)

A smooth tube is constrained to rotate with constant angular velocity in a horizontal plane about a point of itself. A particle is attached to the end of a light elastic string of natural length a, the other end of which is attached to the tube at the centre of rotation. It is found that the particle can rest in relative equilibrium at a distance $2a$ from the centre. Shew that if the particle is released from relative rest at a distance a from the centre, the greatest distance it attains subsequently is $3a$, assuming that the tube is of sufficient length.

Question (1940 STEP I Q405)

A bead moves on a rough wire bent into the shape of a circle of radius a and fixed in a vertical plane. If the bead is projected with speed u from the lowest point and if the coefficient of friction is $\frac{1}{2}$, determine the least value of u for which the reaction does not vanish before the bead has reached the highest point of the wire.

Question (1942 STEP I Q406)

A hollow circular cylinder of internal radius a is fixed with its axis horizontal. A particle is projected from a point on the lowest generator and moves initially on the smooth inner surface of the cylinder in a plane at right angles to the axis. Find the velocity of projection in order that the particle shall leave the cylinder and pass through the axis.

Question (1919 STEP II Q409)

Prove that a particle moving in a plane curve has an acceleration u^2/ρ along the normal inwards, where u is the velocity of the particle and ρ is the radius of curvature of the curve. A heavy particle is attached by a fine string to a fixed point. The breaking tension of the string is four times the weight of the particle. The particle is projected horizontally from its highest position above the fixed point with the velocity that it would acquire in falling through twice the length of the string. Find the inclination of the string to the vertical when it breaks.

Question (1914 STEP III Q411)

A perfectly elastic particle is dropped from a point on a fixed vertical circular hoop, shew that after two rebounds it will rise vertically if

$$2\sin 4\theta = \tan\theta,$$

where θ is the angular distance of the point from the highest point of the hoop.

Question (1922 STEP III Q410)

A particle is moving in a circle of radius r with velocity v. Prove that its acceleration towards the centre is v^2/r. A smooth circular tube is held fixed in a vertical plane. A particle of mass m, which can slide inside the tube, is slightly displaced from rest at the highest point of the tube. Find the pressure between the particle and the tube when it is at an angular distance θ from the highest point of the tube. Also find the vertical component of the acceleration of the particle when $\theta = 120°$.

Question (1925 STEP III Q406)

A uniform rectangular plate $ABCD$ is hinged at the fixed point A and is supported in such a position that AB, one of the longer sides, is horizontal, and AD is vertical. When the plate is released it swings in its own plane about the fixed hinge A and comes to rest with AB vertical. The stiffness of the hinge produces a constant retarding couple during motion. Prove that the plate stays in the new position if

$$\frac{AB}{AD} > 1 + \frac{\pi}{2}.$$

Question (1933 STEP III Q408)

Prove that v^2/ρ is the acceleration along the normal inwards of a point moving with velocity v in a curve, where ρ is the radius of curvature at the point. A circular cylinder of radius a is placed in a fixed position with its axis horizontal on a smooth horizontal plane. A perfectly elastic particle is placed on the highest generator of the cylinder and being slightly displaced slides down the cylinder. Prove that the distance between consecutive points at which it strikes the horizontal plane is $40a\sqrt{2}/27$.

Question (1915 STEP IIII Q408)

A particle is projected along the inner side of a smooth circle of radius a, the velocity at the lowest point being u. Shew that if $u^2 < 5ga$ the particle will leave the circle before arriving at the highest point and will describe a parabola whose latus-rectum is $2(u^2 - 2ga)^3/27g^2a^2$.

Question (1915 STEP IIII Q409)

Find the resultant acceleration of a point which moves in any manner round a circle.

The wheel axles of a motor car are 4 feet long and the height of the C.G. is 2 feet. Find the speed of the car if in going round a level track of 400 feet radius the inner wheels just leave the ground.

Question (1916 STEP IIII Q407)

Two rings of masses M, m $(1 < M/m < 1 + \sqrt{2})$ joined by a light rod of length l can slide on two smooth intersecting rods in a vertical plane, each rod making $45°$ with the vertical. The rod starts from rest in a horizontal position. Shew that, when the velocity of the system again vanishes, the greater mass will have descended through a vertical distance $m(M - m)l/(M^2 + m^2)$.

Question (1934 STEP I Q509)

A particle is suspended from a fixed point by a light inextensible string of length l. If the particle receives a horizontal velocity u, find conditions such that the string shall become slack in the subsequent motion, and prove that in this case the string is slack for a time $\sqrt{\frac{8l}{g}} \sin\phi \sin 2\phi$ where $3\cos\phi = \frac{u^2}{gl} - 2$.

Question (1932 STEP II Q502)

Obtain the expressions $v\frac{dv}{ds}$ and $\frac{v^2}{\rho}$ for the tangential and normal components of the acceleration of a particle which is describing a plane curve. A smooth groove in the form of the catenary $s = c\tan\psi$ is fixed in a vertical plane with the line $\psi = 0$ horizontal and vertex downwards. A particle of mass m starts from rest in the position $\psi = \frac{\pi}{4}$ and moves freely in the groove. Prove that the particle does not leave the groove in the subsequent motion and find the greatest and least values of the reaction between the particle and the groove.

Question (1932 STEP II Q503)

A uniform circular hoop of radius r in a horizontal plane is spinning about its centre with uniform angular velocity 4ω, and is supported by a smooth horizontal table, when a point B of the hoop is momentarily brought to rest. Find the change in the angular velocity of the hoop and shew that the distance between the position of B when brought to rest and its position $\frac{\theta}{\omega}$ units of time later is

$$2r\{\sin^2\theta + \theta^2 - \theta\sin 2\theta\}^{\frac{1}{2}}.$$

Geometry and Trigonometry

In analytical geometry questions it may be assumed that the axes are rectangular.

Question (1918 STEP III Q510)

Prove that v^2/r is the acceleration towards the centre of a particle moving in a circle with velocity v. A heavy particle is placed inside a smooth circular tube fixed in a vertical plane. The particle is slightly displaced from rest at the highest point of the tube, prove that in the subsequent motion the pressures between it and the tube as it passes the extremity of the horizontal diameter and the lowest point of the tube are as 2:5.

Question (1920 STEP III Q506)

A particle is projected along the inner surface of a smooth vertical sphere of radius a, starting at the lowest point A with velocity $\sqrt{\frac{7ag}{2}}$. Prove that the particle will leave the sphere at an angular distance of 60° from the top. Prove also that it will strike the sphere again at its lowest point A.

Question (1921 STEP III Q506)

A body makes complete revolutions about a fixed horizontal axis, about which its radius of gyration is k, and the centre of gravity of the body is at a distance c from the axis. If the greatest and least angular velocities are p per cent. greater and p per cent. less than a quantity ω, prove that

$$\omega = \sqrt{\frac{200gc}{k^2}}.$$

Question (1923 STEP III Q509)

Determine the acceleration of a point describing a circle with uniform speed. A small ring fits loosely on a rough spoke (length a) of a wheel which can turn about a horizontal axle and the ring is originally at rest in contact with the lowest point of the rim: if the wheel is now made to revolve with uniform angular velocity ω, prove that the angle θ through which the wheel will turn before the ring slides is given by the equation

$$g\cos(\theta - \lambda) + a\omega^2 \cos\lambda = 0,$$

where λ is the angle of friction.

Question (1926 STEP III Q508)

Show that the acceleration of a particle along the normal to its path is v^2/ρ, where ρ is the radius of curvature and v the velocity. A bead moves on a smooth parabolic wire whose axis is vertical and vertex upwards. Show that the pressure between the wire and bead varies inversely as ρ.

Question (1927 STEP III Q506)

A string of length $2l$ has its ends attached to two fixed points A, B, where $AB = l$, and A is vertically above B. A bead C of mass m can slide freely on the string and describes a horizontal circle with angular velocity ω about AB. If y is the depth of the plane of the circle below A, show that $y = \frac{l}{2}\left(1 + \frac{4g}{3l\omega^2}\right)$, and find the tension of the string in terms of ω.

Question (1923 STEP IIII Q509)

A particle of mass m is attached by a string to a point on a fixed circular cylinder of radius a whose axis is vertical. The particle is projected with velocity v at right angles to the string along a smooth horizontal plane so that the string winds itself round the cylinder. Shew (i) that the velocity of the particle is constant; (ii) that the tension of the string is inversely proportional to the length which remains straight at any instant; (iii) that if the initial length of the string is l and the breaking tension is T, the string will break when it has turned through an angle $\dfrac{l}{a} - \dfrac{mv^2}{aT}$.

Question (1926 STEP I Q613)

A particle moves in a circle of radius r, and has a velocity v after time t. Prove that it has an acceleration whose resolved parts are v^2/r towards the centre and $\frac{dv}{dt}$ along the tangent. A mass of 1 lb. is attached to the end of a string which is 20 inches long and is tied to a fixed point A. Initially the string is horizontal and the mass is allowed to fall. Determine the tension in the string when the mass is vertically below A. If the string catches against a peg B vertically below A so that the mass begins to describe a circle about B, find the least depth of B below A in order that the mass may describe a complete circle about B.

Question (1915 STEP III Q606)

A particle is describing a circle uniformly; determine the radial force acting on it.

Two particles are connected by a fine string passing through a smooth ring and describe horizontal circles in the same periodic time; shew that the particles are at the same vertical depth below the ring, and find the ratio in which the string is divided by the ring.

Question (1917 STEP III Q604)

Define angular velocity. A circle, centre C, rolls with uniform angular velocity ω on the outside of another equal fixed circle whose centre is O. Prove that the angular velocity of OP, where P is any point on the circumference of the rolling circle, is $\frac{6\sin^2\theta}{1+8\sin^2\theta}\omega$, where 2θ is the angle OCP.

Question (1924 STEP III Q608)

A car takes a banked corner of a racing track at a speed V, the lateral gradient α being designed to reduce the tendency to side-slip to zero for a lower speed U. Show that the coefficient of friction necessary to prevent side-slip for the greater speed V must be at least

$$\frac{(V^2-U^2)\sin\alpha\cos\alpha}{V^2\sin^2\alpha+U^2\cos^2\alpha}.$$

Question (1927 STEP III Q612)

A particle slides, from rest at a depth $r/2$ below the highest point, down the outside of a smooth sphere of radius r; prove that it leaves the sphere at a height $r/3$ above the centre. Shew further that when the particle is at a distance $r\sqrt{2}$ from the vertical diameter of the sphere it is at a depth $4r$ below the centre.

Question (1917 STEP II Q710)

Prove that v^2/ρ is the acceleration inwards along the normal, when a particle describes a plane curve with velocity v. A smooth parabolic tube has its axis vertical and vertex upwards. A particle inside the tube is projected from the vertex with a velocity $\sqrt{2g(a+h)}$. When the particle is at a depth z below the directrix prove that the pressure on the tube is $mgh\sqrt{az^{-3}}$, where m is the mass of the particle and $4a$ is the latus rectum of the parabola.

Question (1921 STEP II Q706)

A sphere is set rolling on a horizontal plane which is made to rotate about a fixed vertical axis with constant angular velocity ω. Prove that with suitable initial conditions the path in space of the centre of the sphere is a circle described with angular velocity

$$\frac{k^2\omega}{a^2+k^2},$$

where a is the radius and k the radius of gyration of the sphere about a diameter.

Question (1920 STEP III Q712)

An infinite circular cylinder of radius b and uniform density σ is surrounded by fluid of density ρ. The outer boundary of the fluid is a concentric circular cylinder of radius a. The outer cylinder is caused to execute small oscillations of amplitude α in a direction perpendicular to its length. Show that the amplitude β of the oscillations of the inner cylinder is

$$\beta = \frac{2a^2\rho}{a^2(\sigma+\rho)+b^2(\sigma-\rho)}.$$

Question (1923 STEP II Q803)

A uniform hollow circular cylinder is free to turn about its axis which is horizontal. A uniform sphere is placed on the top of the cylinder and is slightly disturbed in such a way that its centre moves in a plane perpendicular to the axis of the cylinder in the motion that ensues. Show that slipping will in all cases take place before the sphere leaves contact with the cylinder, and that it commences when

$$2M\sin\theta = \mu[(17M+6m)\cos\theta - (10M+4m)].$$

M, m are the masses of the cylinder and sphere, respectively, μ is the coefficient of friction, and θ the angle a perpendicular from the point of contact on the axis of the cylinder makes with the vertical. It is assumed that the radius of gyration of the cylinder is equal to its radius. What may be deduced from the above equation by writing (i) $M=0$, (ii) $m=0$?

Question (1923 STEP II Q805)

A body is moving, under gravity, in contact with a smooth horizontal plane. Taking as axes of reference the principal axes at the centre of gravity, write down equations sufficient to determine the motion. A uniform prolate spheroid of semi-axes a, c, e, is rotating about its axis of revolution which is vertical with angular velocity ω in contact with a smooth horizontal plane. Shew that the motion is stable if

$$\omega^2 > \frac{5g(a^2-c^2)}{ac^4}.$$

Question (1913 STEP III Q812)

Any point S on a sphere is displaced on the great circle through a fixed point O on the sphere to a point S' by the aberration law $\sin SS' = k \sin SO$, where k is any finite number less than unity. Prove that, if S lie on a great circle having any fixed point P as pole, S' lies on a small circle with P as pole.

Question (1919 STEP III Q810)

A smooth circular cylinder of radius a is placed in a fixed position on a horizontal table. A heavy particle is placed at rest on the highest generator of the cylinder and is slightly displaced. Prove that the particle will strike the table at a distance

$$5(\sqrt{5} + 4\sqrt{2})a/27$$

from the line in which the cylinder touches the table.

Question (1922 STEP III Q804)

A uniform circular hoop of radius r rolls steadily on a horizontal plane so that its centre describes with velocity V a horizontal circle of radius R. Its plane makes a constant angle α with the horizontal. Prove that

$$V^2 = \frac{2gR^2 \cot \alpha}{4R + r \cos \alpha}.$$