Question (1951 STEP I Q101)
(i) Given that α and β are the roots of

$$x^2 - px + q = 0,$$

form the equation whose roots are $\alpha^3 - \frac{1}{\beta^3}$, $\beta^3 - \frac{1}{\alpha^3}$. (ii) Given that the equation

$$x^n - ax^2 + bx - c = 0, \quad (c \neq 0, n > 2),$$

has a thrice repeated root ξ, establish the relations

$$\xi = \frac{(n-1)b}{2(n-2)a} = \frac{2nc}{(n-1)b},$$

and

$$\xi^n = \frac{2c}{(n-1)(n-2)}.$$

Question (1951 STEP I Q409)
Show by comparison with the identity $4\cos^3\alpha - 3\cos\alpha - \cos 3\alpha = 0$ that the cubic equation $x^3 - 3qx - r = 0$ can be solved in terms of cosines provided that $4q^3 > r^2$. If α is defined by the equation $\cos 3\alpha = r/2q^{3/2}$, show that $2q^{1/2}\cos\alpha$ is a root, and find the other two roots. Use the method to solve the equation

$$x^3 - 6x^2 + 6x + 8 = 0.$$

Question (1952 STEP IIII Q101)
Prove that the sum of the roots of the equation

$$\begin{vmatrix} x & h & g \\ h & x & f \\ g & f & x \end{vmatrix} = 0$$

is zero, and that the sum of the squares of the roots is

$$2(f^2 + g^2 + h^2).$$

Taking f, g, h to be real, and assuming that the roots are then all real, prove that no root exceeds

$$2\sqrt{\tfrac{1}{3}(f^2 + g^2 + h^2)}$$

in absolute value. In what circumstances (if any) can a root be equal to this in absolute value?

Question (1951 STEP IIII Q202)

Find the condition on the coefficients p, q, r, s of the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

for two of its roots α, β to satisfy the equation $\alpha + \beta = 0$. Show that the equation

$$x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$$

satisfies this condition, and solve it completely.

Question (1953 STEP IIII Q202)

If the polynomial

$$ax^3 + x^2 - 3bx + 3b^2$$

has two coincident zeros show that, in general, it is a perfect cube. Hence, or otherwise, show that if

$$x^4 + 4ax^3 + 2x^2 - 4bx + 3b^2$$

has three equal zeros then the fourth is identical with them, and find for what values of a and b this is the case.

Suppose $ax^3 + x^2 - 3bx + 3b^2 = a(x-\alpha)^2(x-\beta)$, then if $F(x) = ax^3 + x^2 - 3bx + 3b^2$ we must have $F(\alpha) = F'(\alpha) = 0$,

therefore:

$$\begin{aligned}
0 &= a\alpha^3 + \alpha^2 - 3b\alpha + 3b^2 \\
0 &= 3a\alpha^2 + 2\alpha - 3b \\
\Rightarrow \qquad 3b &= \alpha(3a\alpha + 2) \\
\Rightarrow \qquad 0 &= a\alpha^3 + \alpha^2 - \alpha^2(3a\alpha + 2) + \alpha^2(3a\alpha+2)^2/3 \\
&= \alpha^2((a - 3a)\alpha - 1 + \frac{1}{3}(9a^2\alpha^2 + 12a\alpha + 4)) \\
&= \frac{1}{3}\alpha^2(9a\alpha^2 + 6a\alpha + 1) \\
&= \frac{1}{3}\alpha^2(3a\alpha + 1)^2
\end{aligned}$$

Therefore $\alpha = 0$ or $\alpha = -\frac{1}{3a}$.

If $\alpha = 0$, then $b = 0$ and we don't (necessarily) have cubed roots. But in the general form (say when $b \neq 0$) we must have

$2\alpha + \beta = -\frac{1}{a} \Rightarrow \beta = -\frac{1}{3a} \Rightarrow \beta = \alpha$, therefore we have repeated roots.

Similarly, if $F(x) = x^4 + 4ax^3 + 2x^2 - 4bx + 3b^2$ $F'(x) = 4x^3 + 12ax^2 + 2x - 4b = 12a(\frac{1}{3a}x^3 + x^2 + \frac{1}{6a} - \frac{b}{3a})$ then we must have α is a root of $F''(x) = 12x^2 + 24ax + 4 = 4(3x^2 + 6ax + 1)$

Question (1950 STEP IIII Q301)

The roots of the cubic equation

$$ax^3 + bx^2 + cx + d = 0 \quad (a \neq 0, d \neq 0)$$

are α, β, γ. Find the equations whose roots are (i) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$; (ii) $\alpha^2, \beta^2, \gamma^2$; (iii) $\beta\gamma/\alpha, \gamma\alpha/\beta, \alpha\beta/\gamma$. Deduce a necessary and sufficient condition that the product of two of the roots of the original equation should be equal to the third root.

Question (1951 STEP IIII Q302)

(i) Show that, if

$$x^3 + px + q = 0,$$
$$x^3 + rx + s = 0$$

have a common root, then

$$(q - s)^3 = (ps - qr)(p - r)^2.$$

(ii) If α, β, γ are the roots of $x^3 + px + q = 0$, find the equation with roots $\alpha^3, \beta^3, \gamma^3$.

Question (1950 STEP II Q404)

Prove that if the two equations

$$ax^2 + 2bx + c = 0$$
$$a'x^2 + 2b'x + c' = 0$$

have a single common root, then

$$4(b^2 - ac)(b'^2 - a'c') - (ac' + ca' - 2bb')^2 = 0.$$

Show that the condition

$$4(b^2 - ac)(b'^2 - a'c') - (ac' + ca' - 2bb')^2 \geq 0,$$

is necessary for the fraction

$$\frac{ax^2 + 2bx + c}{a'x^2 + 2b'x + c'}$$

to assume all real values for real values of x, and that if this condition is not fulfilled, the range of inadmissible values of the fraction will either be entirely between or entirely outside the roots of the equation

$$x^2(b'^2 - a'c') + x(ac' + ca' - 2bb') + b^2 - ac = 0.$$

Question (1951 STEP II Q402)

Show that in any algebraic equation

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \cdots + (-1)^n p_n = 0$$

the coefficient p_r is the sum of all the products formed by taking the roots together r at a time. If the roots are $x_1, x_2, \dots, x_n$, prove that

$$(1 - p_2 + p_4 - \dots)^2 + (p_1 - p_3 + p_5 - \dots)^2 = (1 + x_1^2)(1 + x_2^2)\dots(1 + x_n^2).$$

Question (1953 STEP II Q401)

Three roots of the quartic equation

$$(x^2 + 1)^2 = ax(1 - x^2) + b(1 - x^4)$$

satisfy the equation

$$x^3 + px^2 + qx + r = 0.$$

Prove that

$$p^2 - q^2 - r^2 + 1 = 0.$$

Question (1944 STEP I Q101)

Prove that the sum of the roots of the equation

$$\begin{vmatrix} a_1 - x & b_1 & c_1 \\ a_2 & b_2 - x & c_2 \\ a_3 & b_3 & c_3 - x \end{vmatrix} = 0$$

is $a_1 + b_2 + c_3$. Express the sum of the squares of the roots in terms of $a_1, b_1, \dots, c_3$.

Question (1945 STEP I Q102)

Prove that, if $h(x)$ is the H.C.F. of two polynomials $f(x), g(x)$, then polynomials $A(x), B(x)$ exist such that

$$A(x)f(x) + B(x)g(x) = h(x).$$

Obtain an identity of this form when

$$f(x) = x^{10} + 1, \quad g(x) = x^6 + 1.$$

Question (1946 STEP I Q102)

For what values of r does the equation

$$x^3 - 3x + r = 0$$

have three distinct real roots? Solve completely the equation

$$4x^3 - 27a^2(x - a) = 0.$$

Question (1947 STEP I Q101)

Find all the solutions of the equations

$$\begin{aligned} x + 2y + 4z &= 12, \\ xy + 2xz + 4yz &= 22, \\ xyz &= 6. \end{aligned}$$

Question (1948 STEP I Q103)

If a, b and c are the roots of the equation $x^3 = px + q$, express $a^2+b^2+c^2$, $a^3+b^3+c^3$ and $a^4 + b^4 + c^4$ in terms of p and q. If

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix},$$

show that $\Delta = (a - b)(b - c)(c - a)$ and that $\Delta^2 = 4p^3 - 27q^2$. If x, y and z are real numbers, prove that

$$2(x - y)^2(y - z)^2(z - x)^2 \leq [x^2 + y^2 + z^2 - \frac{1}{3}(x + y + z)^2]^3$$

and that the equality sign holds if and only if the three numbers are in arithmetic progression.

Question (1944 STEP IIII Q301)

Show that, if x is a root of the equation $x^4 - 6x^2 + 1 = p(x^3 - x)$, then $\frac{1+x}{1-x}$ is also a root, and find the other two roots in terms of x. Hence show that if p is real all the four roots of the equation are real.

Question (1947 STEP IIII Q301)

Solve:

$$\begin{aligned} x + y + z &= 1, \\ x^2 + y^2 + z^2 &= 21, \\ x^3 + y^3 + z^3 &= 55. \end{aligned}$$

Question (1947 STEP IIII Q302)

The equation $x^4 + ax^3 + bx^2 + cx + d = 0$ is such that the sum of two of its roots is equal to the sum of the remaining two. Shew that $a^3 - 4ab + 8c = 0$.
If, in particular, $a = 2, b = -1, c = -2, d = -3$, find all the roots.

Question (1945 STEP II Q404)

Find the condition that the two equations

$$x^2 + 2ax + b^2 = 0,$$
$$x^3 + 3p^2x + q^3 = 0$$

should have a common root. Verify your condition from first principles

(i) when $a = b$;

(ii) when $q = 0, b \neq 0$.

Question (1947 STEP II Q401)

(i) Given that the product of two of the roots is 2, solve the equation

$$x^4 + 2x^3 - 14x^2 - 11x - 2 = 0.$$

(ii) Show that if a is a root of the equation

$$x^4 + 3x^3 - 6x^2 - 3x + 1 = 0$$

then so also is $\frac{a-1}{a+1}$. Express the remaining roots in terms of a, and hence, or otherwise, solve the equation completely.

Question (1947 STEP II Q402)

Prove that if

$$1 + c_1x + c_2x^2 + c_3x^3 + \cdots = (ax^2 + 2bx + 1)^{-1},$$

then

$$1 + c_1^2x + c_2^2x^2 + c_3^2x^3 + \cdots = \frac{1 + ax}{1 - ax}\{a^2x^2 + 2(a - 2b^2)x + 1\}^{-1}.$$

Suppose the roots of $ax^2 + bx + 1$ are $r_\pm = \frac{-b\pm\sqrt{b^2-a}}{a}$ and note that $r_+r_- = \frac{1}{a}$

$$\begin{aligned}\frac{1}{ax^2+2bx+1} &= \frac{1}{a(x-r_+)(x-r_-)} \\ &= \frac{1}{a(r_+-r_-)}\left(\frac{1}{x-r_+} - \frac{1}{x-r_-}\right) \\ &= \frac{1}{a(r_+-r_-)}\left(\frac{1/r_-}{1-\frac{x}{r_-}} - \frac{1/r_+}{1-\frac{x}{r_+}}\right)\end{aligned}$$

$$= \frac{1}{a(r_+ - r_-)} \sum_{n=0}^{\infty} \left(\frac{x^n}{r_-^{n+1}} - \frac{x^n}{r_+^{n+1}} \right)$$

$$= \frac{1}{a(r_+ - r_-)} \sum_{n=0}^{\infty} \left(\frac{r_+^{n+1} - r_-^{n+1}}{(r_- r_+)^{n+1}} \right) x^n$$

$$= \frac{1}{(r_+ - r_-)} \sum_{n=0}^{\infty} \left(r_+^{n+1} - r_-^{n+1} \right) a^n x^n$$

Therefore,

$$c_n = \left(\frac{r_+^{n+1} - r_-^{n+1}}{r_+ - r_-} \right) a^n$$

$$\Rightarrow \quad c_n^2 = \left(\frac{r_+^{n+1} - r_-^{n+1}}{r_+ - r_-} \right)^2 a^{2n}$$

$$= \left(\frac{r_+^{2n+2} + r_-^{2n+2} - 2(r_+ r_-)^{n+1}}{(r_+ - r_-)^2} \right) a^{2n}$$

$$= \left(\frac{r_+^{2n+2} + r_-^{2n+2} - 2a^{-(n+1)}}{\frac{4(b^2-a)}{a^2}} \right) a^{2n}$$

$$= \frac{a^{2n+2}}{4(b^2 - a)} \left(r_+^{2n+2} + r_-^{2n+2} \right) - \frac{a^{n+1}}{2(b^2 - a)}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} c_n^2 x^n = \frac{a^2}{4(b^2 - a)} \left(\frac{r_+^2}{1 - (a^2 r_+^2 x)} + \frac{r_-^2}{1 - (a^2 r_-^2 x)} \right) - \frac{a}{2(b^2 - a)} \frac{1}{1 - ax}$$

Question (1924 STEP I Q104)

Prove that, if α, β, γ are the roots of

$$x^3 + qx + r = 0,$$

then

$$\alpha^2(\beta + \gamma) + \beta^2(\gamma + \alpha) + \gamma^2(\alpha + \beta) = 3r$$

and

$$\alpha^3(\beta + \gamma) + \beta^3(\gamma + \alpha) + \gamma^3(\alpha + \beta) = -2q^2.$$

Question (1924 STEP I Q109)

A family of parabolas have a given point as vertex, and all pass through another given point. Prove that the locus of their foci is a cubic curve.

Question (1925 STEP I Q104)
If α, β, γ are the roots of

$$x^3 - 6x^2 + 18x - 36 = 0,$$

prove that

$$\alpha^2 + \beta^2 + \gamma^2 = 0,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 0.$$

Question (1933 STEP I Q101)
Shew that, if

$$(b-c)^2(x-a)^2 + (c-a)^2(x-b)^2 + (a-b)^2(x-c)^2 = 0,$$

and no two of a, b, c are equal, then

$$x = \tfrac{1}{3}\{a+b+c \pm (a^2+b^2+c^2-bc-ca-ab)^{\frac{1}{2}}\}.$$

Shew that one root of the equation $x^3 = 100(x-1)$ is approximately 1.0103, and determine the other roots, correct to two places of decimals.

Question (1935 STEP I Q101)
Find for what values of the constant a the equation $x^3 - 3x + a = 0$ has three distinct real roots. Shew that if $h > 0$ the equation $x^3 - 3x - 2 - 27h = 0$ has just one real root, and that, if this is denoted by $2 + 3\xi$, then $0 < \xi < h$; and with the aid of this result obtain the narrower limits

$$\frac{h}{(1+h)^2} < \xi < h.$$

Question (1936 STEP I Q104)
If the roots x_1, x_2, x_3 of the equation

$$x^3 = 3p^2x + q$$

are all real and distinct, prove that $4p^6 > q^2$. Obtain the equation whose roots are $x_1^2 - r, x_2^2 - r, x_3^2 - r$, where $3r = x_1^2 + x_2^2 + x_3^2$. Given $x_1^2 < x_2^2 < x_3^2$, prove that

$$x_2^2 - x_1^2 < x_3^2 - x_2^2$$

if and only if $2p^6 < q^2$.

Question (1937 STEP I Q101)

Show that, if $p \neq 0$ and $4p^3 + 27q^2 \neq 0$, the cubic polynomial $x^3 + px + q$ can be expressed in the form

$$\frac{\alpha(x-\beta)^3 - \beta(x-\alpha)^3}{\alpha - \beta}$$

where α and β are certain constants. Hence, or otherwise, prove that, if $4p^3+27q^2 \neq 0$, the cubic equation $x^3 + px + q = 0$ has three unequal roots, which can be found by solving a quadratic equation and a cubic equation of the special type $x^3 = a$.

Question (1938 STEP I Q104)

Prove that, if the fraction p/q is in its lowest terms, there are exactly q different values of the expression $(\cos\theta + i\sin\theta)^{p/q}$. Prove that the equation whose roots are $\tan(4r+1)\frac{\pi}{20}$, $(r = 0, 1, 2, 3, 4)$, is

$$x^5 - 5x^4 - 10x^3 + 10x^2 + 5x - 1 = 0.$$

Question (1938 STEP I Q106)

If $u_0 = 1$ and $u_n = \dfrac{2u_{n-1}+3}{u_{n-1}+2}$, prove that, as the positive integer n tends to infinity, u_n tends to the limit $\sqrt{3}$. (You may find it useful to prove that ${u_n}^2 < 3$ and that $u_{n+1} > u_n$.)

Question (1939 STEP I Q101)

State (without proof) Descartes' rule of signs connecting the number of positive roots of an algebraic equation with the signs of the coefficients, and deduce a similar rule for the number of negative roots.

Find the numbers of positive and negative roots of the equations

$$x^4 - x^3 + x^2 - 1 = 0,$$
$$x^8 - x^3 - x^2 + 1 = 0.$$

Question (1918 STEP I Q105)

Shew that if the cubic equation derived by clearing of fractions the equation

$$\frac{a}{x+a} + \frac{b}{x+b} = \frac{c}{x+c} + \frac{d}{x+d}$$

has a pair of equal roots, then either one of the numbers a, b is equal to one of the numbers c, d or

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{d}.$$

Question (1914 STEP I Q105)

Show that, if α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ then

$$\alpha + \beta + \gamma = -p, \quad \beta\gamma + \gamma\alpha + \alpha\beta = q, \quad \alpha\beta\gamma = -r.$$

Show that if $\alpha\beta + 1 = 0$ then

$$1 + q + pr + r^2 = 0.$$

Question (1919 STEP I Q104)

Prove that, if S_r denotes $1^r + 2^r + 3^r + \cdots + n^r$, then

$$S_5 + S_7 = 2S_1^2. \quad \textit{[sic]}$$

Question (1920 STEP I Q108)

Prove the identity

$$\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}.$$

Sum the series (n terms)

$$\sin\theta + 2\sin 2\theta + 3\sin 3\theta + \cdots + n\sin n\theta.$$

Question (1918 STEP I Q104)

Solve the equations

$$\begin{aligned} x + y + z &= 5 \\ x^2 + y^2 + z^2 &= 13\frac{1}{2} \\ x^3 + y^3 + z^3 &= 44. \end{aligned}$$

Question (1924 STEP I Q104)

Prove carefully that, if

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m$$

vanishes for m distinct values $x_1, x_2, \dots, x_m$ of x, then

$$f(x) = a_0(x - x_1)(x - x_2)\dots(x - x_m).$$

Shew that

$$\cos n\theta - \cos n\phi = 2^{n-1} \prod_{r=0}^{r=n-1} \left\{\cos\theta - \cos\left(\phi + \frac{2r\pi}{n}\right)\right\},$$

and that

$$\sin n\theta = 2^{n-1} \sin\theta \sin\left(\theta + \frac{\pi}{n}\right) \sin\left(\theta + \frac{2\pi}{n}\right) \dots \sin\left(\theta + \frac{(n-1)\pi}{n}\right).$$

$$\sqrt{n} = 2^{\frac{n-1}{2}} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \dots$$

the last factor being $\sin\frac{(n-2)\pi}{2n}$ or $\sin\frac{(n-1)\pi}{2n}$ according as n is even or odd. Criticise the argument: $\sin\theta$ vanishes for $\theta = \pm r\pi$ and for no other values of θ, therefore

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\dots\left(1 - \frac{\theta^2}{r^2\pi^2}\right)\dots.$$

Question (1942 STEP I Q104)

Prove that the equation whose roots are the cubes of the roots $x_1, x_2, \dots, x_n$ of the equation $a_0 + a_1x + \cdots + a_nx^n = 0$ $(a_n \neq 0)$ is

$$X^3 + xY^3 + x^2Z^3 - 3xXYZ = 0,$$

where

$$X = a_0 + a_3x + a_6x^2 + \dots,$$
$$Y = a_1 + a_4x + a_7x^2 + \dots,$$
$$Z = a_2 + a_5x + a_8x^2 + \dots,$$

the sums continuing as long as the suffixes of the a's do not exceed n. Prove also that, if $a_0 \neq 0$,

$$\prod_{r=1}^{n}(x_r + 1 + x_r^{-1}) = \frac{(-1)^n}{a_0a_n}(A^2 + B^2 + C^2 - BC - CA - AB),$$

where A, B, C are the values of X, Y, Z for $x = 1$.

Question (1920 STEP I Q112)

Find the equation which gives the values of x for which $f(x)$ is stationary, where

$$f(x) = x^3/\{e^{x/T} - 1\};$$

and show that, if x_0 is a root of this equation, then as T varies, x_0/T is constant.

Question (1913 STEP II Q202)

Show that if $x^n + a_1x^{n-1} + \cdots + a_n = 0$, where the a's are rational numbers, then any polynomial in x with rational coefficients can be expressed as a polynomial of degree not greater than $(n-1)$ with rational coefficients. Show that if $x^4 + 4ax^3 + 6bx^2 + 4cx + d = 0$ and $y = x^2 + 2ax$, then y satisfies a quadratic equation provided $c = 3ab - 2a^3$ and that the values of y are real if $c^2 \geq a^3d$.

Question (1914 STEP II Q202)

The quadratic equation $x^2 + 2bx + c$, where $b^2 > c$, has real roots x_1, x_2: form the equation of which the roots are $x_1^2 - a^2, x_2^2 - a^2$ and shew that x_1 and x_2 are both outside the interval $-a$ to $+a$ provided $(c + a^2)^2 > 4a^2b^2 > 2a^2(c + a^2)$. Determine the conditions that the roots of the biquadratic

$$x^4 + 1 + 2p(x^3 + x) + qx^2 = 0$$

may be all real and unequal.

Question (1917 STEP II Q204)

Prove that a simple periodic continued fraction

$$a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots\frac{1}{a_n+}\frac{1}{a_1+}\frac{1}{a_2+}\cdots$$

is the positive root of a quadratic equation with coefficients rational in $a_1, a_2, \ldots a_n$. Shew also that the negative root is $-\left(\frac{1}{a_n+}\frac{1}{a_{n-1}+}\cdots\frac{1}{a_1+}\frac{1}{a_n+}\frac{1}{a_{n-1}+}\cdots\right)$. Prove that the $2n$th convergent of $\frac{1}{1+}\frac{1}{k+}\frac{1}{1+}\frac{1}{1+}\cdots$ is the nth convergent of $\frac{k}{k+1-}\frac{1}{k+2-}\frac{1}{k+2-}\cdots$.

Question (1919 STEP II Q201)

Find the equation whose roots are the squares of the reciprocals of the roots of the equation

$$ax^2 + 2bx + c = 0.$$

Question (1927 STEP II Q202)
Shew that, if $c^2 = a^2 d$, then the product of two of the roots of the equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

is equal to the product of the other two. Hence, or otherwise, solve the equation

$$x^4 + x^3 + 2x^2 + 2x + 4 = 0.$$

Question (1927 STEP II Q204)
Comment on the following statements:

(i) If θ is small, $\sin\theta$ is approximately equal to θ.

(ii) The "sum to infinity" of a series is the sum of n terms when n becomes infinite.

(iii) An equation of the nth degree has n roots.

(iv) $8^{\frac{1}{3}} = 2$.

Explain how it is that any meaning is assigned to such an expression as $8^{\frac{1}{3}}$.

Question (1933 STEP II Q202)
Five numbers x, y, z, b and c are connected by the following three relations:

$$x + y + z = 0,$$
$$x^2 + y^2 + z^2 = b,$$
$$x^3 + y^3 + z^3 = c.$$

Find a relation connecting b and c, the satisfying of which is a necessary and sufficient condition that two of x, y and z shall be equal.

Question (1935 STEP II Q201)
By inspection, or otherwise, find all the real roots of each of the equations

1. $(x-1)^3 + (x-2)^3 = 0$,
2. $(x-1)^4 + (x-2)^4 = 1$,
3. $(x-1)^4 + (x-2)^4 = 0$.

Question (1940 STEP II Q202)

By means of a graph, or otherwise, determine the values of λ for which the equation

$$(x-1)^2(x-a)+\lambda=0$$

has three real roots, where a is a given constant greater than unity.

[If any general formula is quoted, it must be proved.]

Prove that, whatever the values of a, λ, the roots α, β, γ are connected by the relation

$$\beta\gamma+\gamma\alpha+\alpha\beta-2(\alpha+\beta+\gamma)+3=0.$$

Question (1940 STEP II Q203)

(i) Solve the equation

$$x^4-x^3+x^2-x+1=0.$$

(ii) Find, in terms of p and q, the cubic equation such that, if x is any one of its roots, $px+q$ is also a root.

Question (1942 STEP II Q202)

The sum of two roots of the equation

$$x^4-8x^3+19x^2+4\lambda x+2=0$$

is equal to the sum of the other two. Determine the value of λ and solve the equation.

Question (1914 STEP IIII Q203)

Shew that the relation independent of λ, which is satisfied by the roots of the quadratic $az^2+bz+c+\lambda(a'z^2+b'z+c')=0$, is

$$\left(z_1-\frac{\beta}{\gamma}\right)\left(z_2-\frac{\beta}{\gamma}\right)=\frac{\beta^2-\alpha\gamma}{\gamma^2},$$

where $\alpha=bc'-b'c, \beta=ca'-c'a, \gamma=ab'-ba'$. Deduce that, if a,b,c,a',b',c' be real numbers, the roots are real for all real values of λ, provided $\beta^2-\alpha\gamma$ be negative; but that, when $\beta^2-\alpha\gamma$ is positive, there are real values of λ for which the roots are imaginary and that the points representing them in the Argand diagram lie on the circle $\left(x-\frac{\beta}{\gamma}\right)^2+y^2=\frac{\beta^2-\alpha\gamma}{\gamma^2}$.

None

Question (1929 STEP IIII Q205)

Prove that the equations

$$a(x) \equiv a_0x^3 + a_1x^2 + a_2x + a_3 = 0$$
$$\text{and} \quad b(x) \equiv b_0x^3 + b_1x^2 + b_2x + b_3 = 0$$

will have a common root if

$$\Delta \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \end{vmatrix} = 0.$$

If

$$a(x) = (x - \lambda)(a_0'x^2 + a_1'x + a_2')$$
$$b(x) = (x - \lambda)(b_0'x^2 + b_1'x + b_2')$$

and

$$\Delta_1 \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \\ b_0 & b_1 & b_2 & b_3 \end{vmatrix}$$

prove that

$$\Delta_1 \times \begin{vmatrix} 1 & \lambda & \lambda^2 & \lambda^3 \\ 0 & 1 & \lambda & \lambda^2 \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a_0' & a_1' & a_2' & 0 \\ 0 & a_0' & a_1' & a_2' \\ 0 & b_0' & b_1' & b_2' \\ b_0' & b_1' & b_2' & 0 \end{vmatrix}$$

and hence that the equations $a(x) = 0, b(x) = 0$ will have two common roots if $\Delta = 0$ and $\Delta_1 = 0$.

Question (1938 STEP IIII Q204)

Shew how the H.C.F. of two polynomials $f(x)$ and $g(x)$ may be found without solving the equations $f(x) = 0$ and $g(x) = 0$. If $d(f)$ denotes "the degree of $f(x)$," shew that necessary and sufficient conditions that $f(x)$ and $g(x)$ have a common factor are that polynomials $F(x)$ and $G(x)$ exist such that $d(F) \leq d(g) - 1$, $d(G) \leq d(f) - 1$, and $Ff = Gg$. Hence find the values of a for which the equation

$$x^5 + x^2 + x + a = 0$$

has a repeated root.

Question (1942 STEP IIII Q203)

If $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$x^4 + px^2 + qx + r = 0,$$

prove that the equation whose roots are $\beta+\gamma, \gamma+\alpha, \alpha+\beta$ is

$$y^3 + 2py^2 + (p + 2s^2)y - q = 0, \text{ (OCR error, likely } p^2 \text{ or similar)}$$

and deduce that the equation whose roots are $\beta+\gamma, \gamma+\alpha, \alpha+\beta, \alpha+\delta, \beta+\delta, \gamma+\delta$ is

$$z^2(z^2+p)^2 - 4rz^2 - q^2 = 0.$$

The roots of the equation $x^4 + 4ax^3 + 6bx^2 + 4cx + d = 0$ are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. By reducing this equation to the form $y^4 + py^2 + qy + r = 0$, or otherwise, prove that the equation whose roots are

$$\tfrac{1}{4}(\lambda_1+\lambda_4-\lambda_2-\lambda_3)^2, \quad \tfrac{1}{4}(\lambda_2+\lambda_4-\lambda_3-\lambda_1)^2, \quad \tfrac{1}{4}(\lambda_3+\lambda_4-\lambda_1-\lambda_2)^2$$

is

$$z(z+6b-6a^2)^2 - 4(d-4ac+6a^2b-3a^4)z - (4c-12ab+8a^3)^2 = 0.$$

Deduce that, if the roots of the equation

$$x^4 + 4ax^3 + 6a^2x^2 + 4cx + d = 0$$

are all real, then they are all equal.

Question (1915 STEP I Q302)

Having given that

$$x + y + z = a,$$
$$x^2 + y^2 + z^2 = b^2,$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{c},$$

determine $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$ in terms of a, b, c.

Solve the equations

$$\begin{cases} xyz = 1, \\ (x+1)(y+1)(z+1) = 9, \\ (y+z)(z+x)(x+y) = 11\tfrac{1}{4}. \end{cases}$$

Question (1931 STEP I Q301)
Find the real roots of the equations

1. $x^3 - 15x + 30 = 0$;
2. $xy(x + y) = 12x + 3y$,
   $xy(4x + y - xy) = 12(x + y - 3)$.

Question (1931 STEP I Q303)
(i) Prove that all the roots of the equation

$$x^4 - 14x^2 + 24x = k$$

are real if $8 < k < 11$. (ii) Prove that if the product of two roots of the equation

$$x^4 - ax^3 + bx^2 - cx + d = 0$$

is equal to the product of the other two,

$$a^2 d = c^2.$$

Question (1932 STEP I Q302)
Find the relation between p and q necessary in order that the equation $x^3 - px + q = 0$ may be put into the form

$$(x^2 + mx + n)^2 = x^4.$$

Hence or otherwise solve the equation

$$8x^3 - 36x + 27 = 0.$$

Question (1933 STEP I Q301)
If the equation

$$x^5 + 5qx^3 + 5rx^2 + t = 0$$

has two equal roots, prove that either of them is a root of the quadratic

$$3rx^2 - 6q^2 x - 4qr + t = 0.$$

Question (1938 STEP I Q302)
Show that the real cubic equation

$$x^3 + ax^2 + b = 0$$

has three real zeros if and only if

$$27b^2 + 4a^3 b \leq 0.$$

Question (1916 STEP II Q304)

Prove that, having given $c^2 = a^2 d$, the product of a pair of the roots of the equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

is equal to the product of the other pair. Solve the equation

$$x^4 - x^3 - 16x^2 - 2x + 4 = 0.$$

Question (1917 STEP II Q304)

Prove that the equation

$$x^4 + 4rx + 3s = 0$$

has no real roots if $r^4 < s^3$.

Question (1917 STEP II Q309)

Prove that from a given point on a cubic curve four tangents can be drawn to the cubic in addition to the tangent at the point; and that the points of contact lie on a conic which touches the cubic at the given point; also that the curvature of the conic at this point is half that of the cubic at the point.

Question (1918 STEP II Q302)

Shew that the sum of the homogeneous products of a, b, c, of n dimensions is $\Sigma a^{n+2}/(b-a)(c-a)$.

Question (1925 STEP II Q306)

Find the equation of the tangent at a point on the curve $f(x, y) = 0$. If the tangent at P on $y^3 = 3ax^2 - x^3$ meets the curve again at Q, prove that

$$\tan QOx + 2\tan POx = 0,$$

O being the origin. Also show that if the tangent at P is a normal at Q, then P lies on

$$4y(3a - x) = (2a - x)(16a - 5x).$$

Question (1930 STEP II Q304)

If $f(x)$ is an algebraic function, shew that between two consecutive real roots of the equation $f'(x) = 0$ there can at most be only one real root of $f(x) = 0$. Prove that the necessary and sufficient condition for the reality of all three roots of the cubic $x^3 + 3px + q = 0$ is that $4p^3 + q^2 < 0$. Discuss the case $4p^3 + q^2 = 0$. Shew that $x^5 + 3x^2 + x + 2 = 0$ has only one real root, and locate it between two consecutive integers.

Question (1914 STEP III Q302)

If a, b, c, d are in ascending order of magnitude, the equation

$$(x-a)(x-c) = k(x-b)(x-d)$$

has real roots for all values of k.

Question (1919 STEP III Q308)

Find the coordinates of the double point of the cubic whose equation is

$$xy(5x+y-6)+3x+3y-2=0.$$

Write down the equation of the tangents at the double point. Are they real?

Question (1923 STEP III Q307)

Eliminate x, y, z from the equations

$$ax^2+by^2+cz^2 = ax+by+cz = yz+zx+xy = 0$$

and reduce the result to a symmetrical form.

Question (1938 STEP III Q302)

Find the cubic, with unity as the coefficient of the highest term, which has the roots

$$2\cos\frac{2\pi}{7}, \quad 2\cos\frac{4\pi}{7}, \quad 2\cos\frac{6\pi}{7}.$$

Question (1939 STEP III Q302)

Prove that

$$(x^2-1)\prod_{\nu=1}^{n-1}\left(x^2-2x\cos\frac{\pi\nu}{n}+1\right) = x^{2n}-1.$$

Hence or otherwise, prove that

$$\prod_{\nu=1}^{\frac{1}{2}(n-1)}\cos\frac{\pi\nu}{n} = 2^{\frac{1}{2}(1-n)} \quad \text{for } n=3,5,7,\dots,$$

and

$$\prod_{\nu=1}^{\frac{1}{2}n-1}\cos\frac{\pi\nu}{n} = \sqrt{n}\,2^{1-n} \quad \text{for } n=4,6,8,\dots.$$

Question (1942 STEP III Q304)

Prove that

$$\tan^2\frac{\pi}{14}+\tan^2\frac{3\pi}{14}+\tan^2\frac{5\pi}{14}=5,$$

and

$$\cos^2\frac{\pi}{14}\cos^2\frac{3\pi}{14}\cos^2\frac{5\pi}{14}=\frac{7}{64}.$$

Question (1921 STEP IIII Q303)

Form an equation with integer coefficients which has

(i) $\sqrt{2}+\sqrt{3}$,

(ii) $\sqrt{2}+\sqrt{3}+\sqrt{5}$,

for a root. State what the other roots are in each case.

Question (1913 STEP I Q401)

Prove that, if $a+b+c=0$, and no two of a,b,c are equal, constants A,B,C can be found to make the equation

$$A(x-a)^3+B(x-b)^3+C(x-c)^3+x=0$$

true for all values of x, and determine their values.

Question (1916 STEP I Q401)

If α stands for the fifth root of 2, and $x=\alpha+\alpha^4$, prove that

$$x^5=10x^2+10x+6.$$

Question (1918 STEP I Q405)

Prove that, if $bc+ca+ab=0$, then

$$\Sigma a^5=\Sigma(a^2)\{\Sigma(a^3)+2abc\}.$$

Question (1930 STEP I Q401)

(i) Find the real roots of the equation

$$x^8 + 1 + (x+1)^8 = 2(x^2+x+1)^4.$$

(ii) Eliminate x, y, z from the equations:

$$\frac{y}{z} + \frac{z}{y} + \frac{x}{z} = a,$$

$$\frac{z}{x} + \frac{x}{z} + \frac{y}{x} = b,$$

$$\left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right) = c.$$

Question (1921 STEP II Q401)

Shew that the roots of

$$(x-b)(x-c) + (x-c)(x-a) + (x-a)(x-b) = 0$$

are real, and cannot be equal unless $a = b = c$. Eliminate x, y from

$$x(x-y) = a^2, \quad y(x+y) = b^2, \quad 1/x^2 + 1/y^2 = 1/c^2.$$

Question (1933 STEP II Q403)

The roots of the equation

$$x^3 + 3px + q = 0$$

are α, β, γ. Find the equation whose roots are $(\beta-\gamma)^2, (\gamma-\alpha)^2, (\alpha-\beta)^2$. Deduce that if

$$4p^3 + q^2 > 0,$$

the original equation has one real and two imaginary roots. Prove also that if a, b, c are the roots of the above equation of squared differences,

$$a^2 + b^2 + c^2 = 2(bc + ca + ab).$$

Question (1942 STEP II Q409)

Prove that the roots of the equation

$$x^4 - x^3\left(4R + 2\frac{\Delta}{s}\right) + x^2 s^2 + x^2\frac{\Delta}{s}(4R + \frac{\Delta}{s}) - 2s\Delta x + \Delta^2 = 0$$

are the radii of the inscribed and three escribed circles of a triangle whose area is Δ, circumradius R, and the sum of whose sides is $2s$.

Question (1916 STEP III Q401)

If α is a root of $ax^2 + 2bx + c = 0$ and β a root of $a'x^2 + 2b'x + c' = 0$, find the equation whose roots are the different values of α/β.

Question (1917 STEP III Q401)

Prove that if $a, b, c, \ldots$ be any number of quantities, $\Sigma a^3 - 3\Sigma abc$ is divisible by Σa, and find the quotient.

Question (1918 STEP III Q402)

Prove that if $(x-b)(x-c) + (x-c)(x-a) + (x-a)(x-b)$ be a perfect square in x, then $a = b = c$. Determine λ so that

$$(3x + 2y - 1)(2x + 3y - 1) + \lambda(x + 4y - 1)(4x + y - 1) = 0$$

may be the product of linear factors.

Question (1937 STEP III Q406)

Shew that the number of real roots of the algebraic equation $f(x) = 0$ cannot exceed by more than unity that of its derived equation $f'(x) = 0$. Find the necessary and sufficient condition for the cubic equation $x^3 + 3ax + b = 0$ to have three real roots. a and b are real.

Question (1938 STEP III Q401)

Shew that there is a unique value of λ for which $ax^4 + 6cx^2 + 4dx + e$ is expressible in the form $A(x-\alpha)^4 + B(x-\beta)^4$, where $\lambda, A, B, \alpha, \beta$ are independent of x. If a, c, d, e are real, find the condition that A, B, α, β shall be real.

Question (1939 STEP III Q405)

If $u_{n+1} = \frac{1}{2}(u_n + 1/u_n)$, and if u_1 is positive, shew that, for $n > 1$,

$$1 \leq u_n,$$

$$u_{n+1} \leq u_n,$$

also that

$$(u_{n+1} - 1) \leq \frac{1}{2}(u_n - 1)^2;$$

$$\frac{1}{2} \leq u_n - u_{n+1}, \quad \text{if } u_n \geq 2.$$

Hence shew that u_n tends to a limit as n tends to infinity, and state the value of the limit.

Question (1942 STEP III Q404)
Show that the cubic equation $x^3 + 3px + q = 0$ can be expressed in the form $a(x+b)^3 - b(x+a)^3 = 0$ by proper choice of a and b. Hence solve the equation $x^3 - 9x + 28 = 0$.

Question (1913 STEP II Q501)
If

$$\begin{aligned} a(x+y+b) + x^2y^2 + bxy(x+y) &= 0, \\ a(z+x+b) + z^2x^2 + bzx(z+x) &= 0, \end{aligned}$$

and y and z are unequal, prove that

$$a(y+z+b) + y^2z^2 + byz(y+z) = 0.$$

Question (1915 STEP II Q501)
Express $\tan n\theta$ in powers of $\tan\theta$, distinguishing the cases according as n is odd or even.

Prove that

$$\tan\theta + \tan\left(\theta + \frac{\pi}{2n}\right) + \tan\left(\theta + \frac{2\pi}{2n}\right) + \cdots + \tan\left(\theta + \frac{2n-1}{2n}\pi\right) = -2n\cot 2n\theta,$$

and find the sum of the squares of the same tangents.

Question (1917 STEP II Q503)
Resolve the expression $x^{2n} - 2x^n\cos n\theta + 1$ into n real quadratic factors, and deduce the factors of $\cos n\phi - \cos n\theta$ regarded as a function of $\cos\phi$.

Question (1919 STEP II Q501)
Prove that, if

$$a + b + c = 0,$$

then

$$a^3 + b^3 + c^3 = 3abc,$$

and

$$a^6 + b^6 + c^6 = \frac{1}{4}(a^2+b^2+c^2)^3 + 3a^2b^2c^2.$$

Solve the equations

$$\begin{aligned} x + y + z &= 4, \\ x^2 + y^2 + z^2 &= 66, \\ x^3 + y^3 + z^3 &= 280. \end{aligned}$$

Question (1921 STEP II Q501)

Solve the equations

$$\begin{aligned} x^2 + 2yz &= -11, \\ y^2 + 2zx &= -2, \\ z^2 + 2xy &= 13. \end{aligned}$$

Given that $x = a$ is one root of the equation

$$(x^2 + x + 1)^3 = M(x^2 + x)^2,$$

prove that the other roots are

$$\frac{1}{a}, \quad -(a+1), \quad -\frac{1}{a+1}, \quad -\frac{a}{a+1}, \quad -\frac{a+1}{a}.$$

Question (1924 STEP II Q504)

Prove that, if the roots of the equation

$$x^3 + px + q = 0$$

are all real, then $4p^3 + 27q^2$ is negative. If the roots are α, β, γ, prove that the value of

$$\Sigma(\beta - \gamma)^3(\beta + \gamma - 2\alpha)$$

is $-27q$.

Question (1916 STEP III Q508)

If $\sqrt{\frac{a}{x-a}} + \sqrt{\frac{b}{x-b}} + \sqrt{\frac{c}{x-c}} = \sqrt{\frac{abc}{(x-a)(x-b)(x-c)}}$, prove that

$$\frac{4abc}{x} = 2bc + 2ca + 2ab - a^2 - b^2 - c^2.$$

Question (1917 STEP III Q506)

Prove that the product of the infinite periodic continued fractions

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+\dots}\,\frac{1}{a_n+}\,\frac{1}{a_1+}\,\frac{1}{a_2+\dots}$$

and

$$\frac{1}{a_n+}\,\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+\dots}\,\frac{1}{a_1+}\,\frac{1}{a_n+}\,\frac{1}{a_{n-1}+\dots}$$

is p_n/q_{n-1} where p_r/q_r is the rth convergent of

$$\frac{1}{a_1+}\,\frac{1}{a_2+\dots}\,\frac{1}{a_n}.$$

Question (1931 STEP III Q501)

Shew that the product $(a^3 + b^3 + c^3 - 3abc)(x^3 + y^3 + z^3 - 3xyz)$ can be expressed in the form $A^3 + B^3 + C^3 - 3ABC$, all the quantities involved being real. Find A, B, C in terms of a, b, c, x, y, z. Shew also that if n is a positive integer of the form $3m + 1$ (m being any positive integer), then $(y - z)^n + (z - x)^n + (x - y)^n$ has a factor $\Sigma a^2 - \Sigma yz$.

Question (1932 STEP III Q501)

Find the equation whose roots are the squares of the differences of the roots taken in pairs of the cubic equation $x^3 + bx + c = 0$. Hence or otherwise shew that if b and c are real the equation $x^3 + bx + c = 0$ will have three real roots or only one real root according as $4b^3 + 27c^2$ is negative or positive. Consider also the case in which $4b^3 + 27c^2 = 0$.

Question (1915 STEP IIII Q504)

Shew that every mixed periodic continued fraction, which has more than one non-periodic element, is a root of a quadratic equation with rational coefficients whose roots are both of the same sign.

Find the value of the $2n$th convergent to the continued fraction

$$\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{4+}\cdots.$$

Question (1927 STEP I Q608)

The equation $4x^5 - 57x^3 + 64x^2 + 108x - 144 = 0$ has two roots which are equal in magnitude and opposite in sign. Solve it completely.

Question (1924 STEP II Q603)

Prove that the continued fraction $a - \frac{1}{a-}\,\frac{1}{a-\ldots}$ in which a is equal to -1 and is repeated any number of times, must have one of three values, and that if a satisfies the equation $2a^3 + 3a^2 - 3a - 2 = 0$, the fraction satisfies this equation.

Question (1925 STEP III Q603)

Prove that $a + b + c + d$ is a factor of the expression

$$2(a^4 + b^4 + c^4 + d^4) - (a^2 + b^2 + c^2 + d^2)^2 + 8abcd.$$

Shew that $a + b - c - d$ is also a factor, and find the remaining factors.

Question (1930 STEP III Q601)

Explain what is meant by a recurring series and define the scale of relation of such a series. How may the sum to n terms of the series be found? Sum to n terms the series:

(i) $1^3 + 2^3.x + 3^3.x^2 + \dots;$

(ii) $1 + 2^2 + 3^3 + 4 + 5^2 + 6^3 + 7 + 8^2 + 9^3 + \dots.$

Question (1919 STEP II Q702)

Find the condition that the equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ should have a common root. If α is a root of the first and α' of the second, find the equation whose roots are the four values of $\alpha - \alpha'$.

Question (1922 STEP III Q703)

If α, β are the roots of the quadratic

$$ax^2 + 2hx + b + \kappa(a'x^2 + 2h'x + b') = 0,$$

prove that numbers p, q, independent of κ, can be found such that $(p-\alpha)(p-\beta) = q^2$, and that $(p \pm q)$ are the roots of the quadratic

$$(ax + h)(h'x + b') - (hx + b)(a'x + h') = 0.$$

Taking x as the abscissa of any point, give a geometrical interpretation of the preceding result.

Question (1922 STEP III Q704)

Prove that

$$\frac{nx^{2n-1}}{x^{2n} - 1} = \frac{x}{x^2 - 1} + \sum_{r=1}^{n-1} \frac{x - \cos r\alpha}{x^2 - 2x\cos r\alpha + 1},$$

where $\alpha = \pi/n$; and deduce that

$$\sum_{r=1}^{n-1} \frac{\cos r\alpha}{\cos r\alpha - \cos\theta} = \frac{n\cos(n-1)\theta}{\sin\theta\sin n\theta} - \csc^2\theta.$$

Question (1924 STEP III Q708)

Prove that the tangents to a parabola at any three points P, Q, R form a triangle whose area is half the area of the triangle PQR.

Question (1919 STEP II Q804)

Resolve $x^{2n} - 2x^n y^n \cos n\theta + y^{2n}$ into factors. Prove that

$$\sin n\phi = 2^{n-1} \sin\phi \sin\left(\phi + \frac{\pi}{n}\right) \dots \sin\left(\phi + \frac{n-1}{n}\pi\right).$$

Question (1919 STEP II Q805)

Prove that the equation of the straight line joining the feet of the perpendiculars from the point (h, k) on the lines $ax^2 + 2bxy + cy^2 = 0$ is

$$x(ah - ch + 2bk) + y(ck - ak + 2bh) + ak^2 - 2bhk + ch^2 = 0.$$