Question (1983 STEP I Q3) (i) Show that a necessary condition for the lines

$$\mathbf{r} = \mathbf{a} + s\mathbf{m}, \quad \mathbf{r} = \mathbf{b} + t\mathbf{n}$$

to intersect is $[(\mathbf{a} - \mathbf{b}), \mathbf{m}, \mathbf{n}] = 0$, where $[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ denotes the scalar triple product $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$ of the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Is the condition $[(\mathbf{a} - \mathbf{b}), \mathbf{m}, \mathbf{n}] = 0$ sufficient for the two lines to intersect?

(ii) Find the points of intersection of the line $\mathbf{r} = \mathbf{a} + s\mathbf{m}$ with the plane $\mathbf{r} \cdot \mathbf{n} = d$, discussing carefully the case $\mathbf{m} \cdot \mathbf{n} = 0$.

Question (1966 STEP II Q9)

Define the curvature κ at a point of a curve having a smoothly-turning tangent. Show that, if the rectangular Cartesian coordinates (x, y) of the general point of the curve are given as functions of a parameter θ, then

$$\kappa = \frac{x'y'' - y'x''}{[(x')^2 + (y')^2]^{3/2}},$$

where dashes denote differentiations with respect to θ. Find the curvature at the general point of the curve

$$x = a\cos\theta + a\theta\sin\theta, \quad (1)$$
$$y = a\sin\theta - a\theta\cos\theta. \quad (2)$$

Verify your result by showing that the normal at any point of the curve touches a circle $x^2 + y^2 = a^2$. Deduce a mechanical method of drawing the curve, and sketch that part corresponding to values of θ in the range $[0, 2\pi]$.

Question (1972 STEP II Q3)

A curve is given parametrically by

$$x = a(\cos\theta + \log\tan\tfrac{1}{2}\theta)$$
$$y = a\sin\theta,$$

where $0 < \theta < \frac{1}{2}\pi$ and a is constant. The points with parameters $\theta, \frac{1}{2}\pi$ are denoted by P, A respectively; the tangent at P meets the x-axis at Q. Prove that $PQ = a$. Let C be the centre of curvature at P and let s be the arc length from A to P. By considering $ds/d\theta$, or otherwise, show that CQ is parallel to the y-axis.

Question (1974 STEP II Q5)

A curve in the Cartesian plane goes through the origin, touching the x-axis there; at any point the product of its radius of curvature R and its arc-length s (measured from O) is a constant, a^2. Obtain the intrinsic equation of the curve and deduce that it may be parametrized thus:

$$\begin{cases} dx = a(2\psi)^{-\frac{1}{2}}\cos\psi d\psi, \\ dy = a(2\psi)^{-\frac{1}{2}}\sin\psi d\psi. \end{cases}$$

Draw a rough sketch of the curve. [You may assume if you wish that

$$\int_0^\infty \frac{\cos\psi}{\sqrt{\psi}}d\psi = \int_0^\infty \frac{\sin\psi}{\sqrt{\psi}}d\psi = \sqrt{\frac{\pi}{2}}.$$

]

Question (1975 STEP II Q3)

Find the surface area of each of the two spheroids that are obtained by rotating the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (b < a),$$

about its major and minor axes. Express the areas in terms of a and the eccentricity e of the ellipse. In each case verify that the limit of the area, as $e \to 0$, is $4\pi a^2$.

Question (1976 STEP II Q1)

Let C be the arc of the parabola $y = \frac{1}{2}x^2$ between $x = 0$ and $x = a$. Calculate the length of C and the area swept out when C is rotated about the x-axis.

Question (1971 STEP III Q7)

P is a variable point on a plane curve Γ, and R is the centre of curvature of Γ at P. Let Δ be the locus of Q, where Q is the mid-point of PR. Show that if ϕ is the angle between the tangent to Γ at P and the tangent to Δ at Q then

$$\tan\phi = \frac{d\rho}{ds},$$

where $\rho = PR$ and s is the arc length of Γ. Prove that if Γ is defined by the equation $\rho^2 + s^2 = a^2$, then Δ is a straight line.

Question (1971 STEP III Q9)

The points O, A, B, C are not coplanar, and the position vectors of A, B, C with respect to O as origin are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. If $\mathbf{p}$ is any vector, show that

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{p} = (\mathbf{a} \cdot \mathbf{p})\mathbf{b} \times \mathbf{c} + (\mathbf{b} \cdot \mathbf{p})\mathbf{c} \times \mathbf{a} + (\mathbf{c} \cdot \mathbf{p})\mathbf{a} \times \mathbf{b}.$$

X, Y, Z are such that X is the centre of the sphere through O, A, B, C; Y is the centre of a sphere which touches the lines OA, OB, OC; and Z is the second common point of the spheres through O with centres A, B and C. Show that the position vectors of X, Y, Z are of the form $\mathbf{x}$, $\lambda\mathbf{y}$, $\mu\mathbf{z}$ respectively, where

$$\begin{aligned} 2[\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{x} &= |\mathbf{a}|^2\mathbf{b} \times \mathbf{c} + |\mathbf{b}|^2\mathbf{c} \times \mathbf{a} + |\mathbf{c}|^2\mathbf{a} \times \mathbf{b} \\ \mathbf{y} &= |\mathbf{a}|\mathbf{b} \times \mathbf{c} + |\mathbf{b}|\mathbf{c} \times \mathbf{a} + |\mathbf{c}|\mathbf{a} \times \mathbf{b} \\ \mathbf{z} &= \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} \end{aligned}$$

and

$$\mu = \frac{2[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{|\mathbf{z}|^2}.$$

Question (1974 STEP III Q16)

Define the vector product of two vectors $\mathbf{x}$ and $\mathbf{y}$. Let $\mathbf{u}$ be a vector of unit length in 3-dimensional space and let $\mathbf{s}$ be a vector perpendicular to $\mathbf{u}$; $\mathbf{s}'$ is the vector obtained by rotating $\mathbf{s}$ through an angle θ about $\mathbf{u}$. Show that, with a suitable sign convention for θ,

$$\mathbf{s}' = \cos\theta\mathbf{s} + \sin\theta(\mathbf{u} \times \mathbf{s}).$$

Now let $\mathbf{r}$ be any vector, and let $\mathbf{r}'$ be the vector obtained by rotating $\mathbf{r}$ through an angle θ about $\mathbf{u}$. Deduce a formula for $\mathbf{r}'$ in terms of $\mathbf{r}, \mathbf{u}$ and θ.

Question (1975 STEP III Q6)

Prove that a curve in the plane has constant curvature $c \neq 0$ if and only if it is a circle (or portion thereof).

Question (1976 STEP III Q16)

Show that $|\mathbf{a} \wedge \mathbf{b}|^2 = a^2b^2 - (\mathbf{a} \cdot \mathbf{b})^2$. If $\mathbf{a} \wedge \mathbf{b} \neq 0$, and if

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{a} \wedge \mathbf{b} = \alpha'\mathbf{a} + \beta'\mathbf{b} + \gamma'\mathbf{a} \wedge \mathbf{b},$$

show that $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$. For some λ, and for some non-zero $\mathbf{x}$,

$$(\mathbf{a} \cdot \mathbf{x})\mathbf{a} + (\mathbf{b} \cdot \mathbf{x})\mathbf{b} = \lambda\mathbf{x}.$$

By looking for solutions of the form $\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{a} \wedge \mathbf{b}$, or otherwise, show that either $\lambda = 0$ or

$$\lambda^2 - (a^2 + b^2)\lambda + |\mathbf{a} \wedge \mathbf{b}|^2 = 0.$$

Question (1978 STEP III Q15)

Two particles of equal mass collide. Before the impact, their velocities are $\mathbf{v}_1$ and $\mathbf{v}_2$ and afterwards they are $\mathbf{v}_1'$ and $\mathbf{v}_2'$. Momentum and energy are conserved. Show that

1. the relative velocity of the particles has the same magnitude before and after the impact;
2. if one particle is initially at rest, the directions of motion after impact are perpendicular to one another.

Now consider the collision of an election of mass m and velocity $\mathbf{v}_0$ with a stationary atom of mass M. Momentum is conserved, but an amount W of kinetic energy is lost. Find a quadratic equation satisfied by the magnitude of $\mathbf{V}$, the velocity of the atom after the collision, and involving the angle between $\mathbf{V}$ and $\mathbf{v}_0$, and deduce that the initial kinetic energy of the election must not be less than $W(1 + (m/M))$

Question (1980 STEP III Q8)

For a curve defined parametrically by functions $x(t)$, $y(t)$, the radius of curvature is given by

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{(\dot{x}\ddot{y} - \ddot{x}\dot{y})}.$$

An ellipse is given by

$$x = a\cos t, \ \ y = b\sin t.$$

Find the parametric equations of the centre of curvature of the ellipse, and sketch its locus. Describe the shape carefully near the points corresponding to $t = 0$, $\pi/2$, π, $3\pi/2$.

Question (1984 STEP III Q5)

A tetrahedron has vertices at the origin, and at points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. The inscribed sphere lies inside the tetrahedron and touches all four faces. Show that this sphere has radius

$$\frac{|[\mathbf{a},\mathbf{b},\mathbf{c}]|}{|\mathbf{a}\times\mathbf{b}| + |\mathbf{b}\times\mathbf{c}| + |\mathbf{c}\times\mathbf{a}| + |(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a})|}$$

where $[\mathbf{a},\mathbf{b},\mathbf{c}]$ denotes the scalar triple product of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

Question (1984 STEP III Q13)

A particle at position $\mathbf{r}(t)$ is subject to a force $\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{H}$ per unit mass, where $\mathbf{E}$ and $\mathbf{H}$ are constant unit vectors and $\mathbf{E} \times \mathbf{H} \neq \mathbf{0}$. If

$$\mathbf{r} = \alpha\mathbf{E} + \beta\mathbf{H} + \gamma\mathbf{E} \times \mathbf{H},$$

derive the differential equations that must be satisfied by $\alpha(t)$, $\beta(t)$ and $\gamma(t)$. The particle starts from the origin at time $t = 0$ with $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 1$. Show that the subsequent motion is given by

$$\alpha = \sin t$$
$$\beta = (1 + \mu)t + \frac{1}{2}\mu t^2 - \mu \sin t$$
$$\gamma = 1 + t - \cos t$$

where

$$\mu = \mathbf{E}.\mathbf{H}.$$

Question (1968 STEP IIII Q13)

In three-dimensional Euclidean space, $\mathbf{u}$ is a fixed vector of unit length, and $\mathbf{r}$ is a given vector. Using the notation of scalar and vector products, show how to write the sum of a part parallel to $\mathbf{u}$ and a part perpendicular to $\mathbf{u}$. Hence, or otherwise, show that if the plane containing $\mathbf{r}$ and $\mathbf{u}$ is rotated through an angle ϕ measured in the clockwise sense relative to the direction of $\mathbf{u}$, and $\mathbf{r}$ is thereby transported to a new position $\mathbf{r}'$, then

$$\mathbf{r}' = \mathbf{r}\cos\phi + \mathbf{u}(\mathbf{r}\cdot\mathbf{u})(1 - \cos\phi) + (\mathbf{u} \times \mathbf{r})\sin\phi.$$

Question (1974 STEP IIII Q4)

C is a closed, differentiable curve which is convex (i.e. any chord cuts it only twice). Points P and P' move round C in an anti-clockwise sense in such a way that the chord PP' has fixed length $2a$; see Fig. 1. Show that the following properties are equivalent, in the sense that if C has any one of them it has all of them:

(i) PP' cuts off a 'segment' S of constant area;

(ii) the tangents to C at P and P' make equal angles with PP';

(iii) PP' touches its envelope at its mid-point M.

[DIAGRAM MISSING] Show further that the curve Γ defined below possesses one (and hence all) of the above properties. Δ is a three-cusped curve as shown (Fig. 2) with cusps of zero angle. Points R, R' are taken a large fixed distance a from Q in either direction along the tangent at the variable point Q of Δ. Then the complete locus of R and R' as Q moves round all of Δ is defined to be Γ. (Δ and a may be assumed chosen so as to make Γ differentiable and convex.)

Question (1976 STEP IIII Q7)

Let S be the surface of a sphere of unit radius. The intersection of S with a plane through its centre is called a great circle. Let Δ be a curvilinear triangle on S whose edges are arcs of great circles C_1, C_2, C_3. By considering the areas of all the regions into which C_1, C_2, C_3 divide S, or otherwise, show that the sum of the angles of Δ is $\pi +$ area of Δ. A convex polyhedron with triangular faces has v vertices, e edges and f faces. Show that $e = \frac{3f}{2}$ and $v - e + f = 2$.

Question (1976 STEP IIII Q16)

An operator T_a on a vector $\mathbf{b}$ is defined by

$$T_a\mathbf{b} = \mathbf{a} \wedge \mathbf{b}.$$

Show that $T_a^3\mathbf{b} = -a^2 T_a\mathbf{b}$. If S_a is defined by

$$S_a\mathbf{b} = (1 + T_a/1! + T_a^2/2! + ...)\mathbf{b},$$

show that

$$S_a\mathbf{b} = \frac{1}{a^2}[(\mathbf{a}\cdot\mathbf{b})\mathbf{a} + \mathbf{a}\wedge\mathbf{b}\sin a - \mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{b})\cos a],$$

and that $|S_a\mathbf{b}|^2 = b^2$.

Question (1977 STEP IIII Q16)

Show that $(\mathbf{l}\wedge\mathbf{m}).\mathbf{n} = (\mathbf{n}\wedge\mathbf{l}).\mathbf{m} = (\mathbf{m}\wedge\mathbf{n}).\mathbf{l}$. Hence, or otherwise, show that

$$|\mathbf{l}\wedge\mathbf{m}|^2 = |\mathbf{l}|^2|\mathbf{m}|^2 - (\mathbf{l}.\mathbf{m})^2.$$

If the point P has position vector $\mathbf{r}$ given by

$$\mathbf{r} = \mathbf{a} + s\mathbf{u}$$

show that P lies on a line if s is allowed to vary, and explain the geometrical significance of $\mathbf{a}$ and $\mathbf{u}$. Suppose two lines are given by equations

$$\mathbf{r}_i = \mathbf{a}_i + s_i\mathbf{u}_i, \quad i = 1, 2.$$

By considering $|(\mathbf{r}_1 - \mathbf{r}_2)\wedge(\mathbf{u}_1\wedge\mathbf{u}_2)|^2$, determine necessary and sufficient conditions for the lines to meet, and if they do not meet, find the shortest distance between them in the two cases $\mathbf{u}_1\wedge\mathbf{u}_2 = \mathbf{0}$ and $\mathbf{u}_1\wedge\mathbf{u}_2 \neq \mathbf{0}$.

Question (1978 STEP III Q15)

Define the scalar product $\mathbf{a} \cdot \mathbf{b}$ and the vector product $\mathbf{a} \wedge \mathbf{b}$ of two vectors. Prove that

$$(\mathbf{a} + \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{c}.$$

Given three non-coplanar vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ prove that an arbitrary vector $\mathbf{x}$ may be written in the form

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{a}^*)\mathbf{a} + (\mathbf{x} \cdot \mathbf{b}^*)\mathbf{b} + (\mathbf{x} \cdot \mathbf{c}^*)\mathbf{c},$$

where

$$\mathbf{a}^* = \frac{\mathbf{b} \wedge \mathbf{c}}{\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})}$$

and $\mathbf{b}^*$, $\mathbf{c}^*$ are defined similarly. Show that $\mathbf{a} = \mathbf{a}^*$, $\mathbf{b} = \mathbf{b}^*$, $\mathbf{c} = \mathbf{c}^*$ if and only if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form an orthogonal triad of unit vectors.

Question (1979 STEP III Q5)

The equation of the tangent plane to the real ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at the point (x_1, y_1, z_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1.$$

Prove that the common tangent planes to the three ellipsoids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\frac{x^2}{b^2} + \frac{y^2}{c^2} + \frac{z^2}{a^2} = 1,$$

$$\frac{x^2}{c^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

touch a sphere of radius $\{(a^2 + b^2 + c^2)/3\}^{\frac{1}{2}}$, and that the points of contact of these planes with the ellipsoids lie on a sphere of radius $(a^4 + b^4 + c^4)^{\frac{1}{2}}(a^2 + b^2 + c^2)^{-\frac{1}{2}}$.

Question (1979 STEP IIII Q16)

Show that for three vectors **a**, **b** and **c**

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{a} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c})$$

and

$$(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{a} \wedge \mathbf{c}) = \mathbf{a}(\mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c}).$$

[You may assume

$$\mathbf{x} \wedge \mathbf{y} \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{y} \wedge \mathbf{z}$$

and

$$(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z} = \mathbf{y}(\mathbf{x} \cdot \mathbf{z}) - \mathbf{x}(\mathbf{y} \cdot \mathbf{z}).$$

Three points A, B and C lie on a sphere with centre O. Let $\hat{A}$, $\hat{B}$ and $\hat{C}$ be the angles BOC, COA and AOB, and let α, β and γ be the angles between the pairs of planes AOB & AOC, BOC & BOA and COA & COB. Deduce the spherical triangle cosine and sine formulae

$$\cos \hat{A} = \cos \hat{B} \cos \hat{C} + \sin \hat{B} \sin \hat{C} \cos \alpha$$

and

$$\frac{\sin \alpha}{\sin \hat{A}} = \frac{\sin \beta}{\sin \hat{B}} = \frac{\sin \gamma}{\sin \hat{C}}.$$

Question (1981 STEP IIII Q12)

A particle of mass m and charge e moves in a constant uniform magnetic field $\mathbf{B}$, so that the force on the particle is $e\mathbf{v} \times \mathbf{B}$ when the particle's velocity is $\mathbf{v}$. Show that: (i) the speed of the particle, $v = |\mathbf{v}|$, is constant; (ii) if at a certain time the particle's velocity is perpendicular to $\mathbf{B}$ then it remains so; (iii) a circular orbit with speed v is possible, and find its radius. Describe the orbit of the particle for general initial conditions.

Question (1982 STEP IIII Q16)

(i) Prove that

$$\frac{d}{dt}\left(\frac{\mathbf{u}}{|\mathbf{u}|}\right) = \frac{1}{|\mathbf{u}|^3}\left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right) \times \mathbf{u},$$

where $\mathbf{u}$ is any function of t. (ii) A particle P of unit mass is acted on by a force of magnitude $\mu/|r|^2$ directed towards a fixed point O, where μ is a constant and $\mathbf{r} = \overrightarrow{OP}$. Its equation of motion is

$$\frac{d\mathbf{v}}{dt} = -\frac{\mu}{|r|^3}\mathbf{r},$$

where $\mathbf{v} = d\mathbf{r}/dt$. By taking an appropriate product with this equation and integrating, prove that $\mathbf{r} \times \mathbf{v}$ is a constant vector $\mathbf{h}$ and deduce that, if $\mathbf{h}$ is non-zero, the motion of P is confined to the plane through O perpendicular to $\mathbf{h}$. Show that

$$\mu\frac{d}{dt}\left(\frac{\mathbf{r}}{|r|}\right) = \frac{d\mathbf{v}}{dt} \times \mathbf{h}$$

and hence by integration show also that

$$\mu(\mathbf{a} \cdot \mathbf{r} + |r|) = |\mathbf{h}|^2$$

for some constant vector $\mathbf{a}$. Relate the magnitude and direction of $\mathbf{a}$ to the geometry of the orbit of the particle.

Question (1964 STEP I Q110)

$p(\phi)$ is the positive length of the projection of a fixed line-segment of length l on an axis at a variable direction ϕ. Prove that

$$\int_{\phi=0}^{\phi=2\pi} p(\phi)d\phi = 4l.$$

Hence or otherwise prove that if a triangle ABC lies entirely within a triangle XYZ, then

$$\text{perimeter } \triangle ABC < \text{perimeter } \triangle XYZ.$$

Question (1964 STEP I Q210)

Prove that the surface area of the spheroid, formed by rotating the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x-axis, is

$$2\pi b^2\left[1 + \frac{a^2}{bc}\sin^{-1}\frac{c}{a}\right],$$

where $c^2 = a^2 - b^2$.

Question (1959 STEP I Q310)

A circle of radius r rolls completely round the outside of a closed convex curve $\mathscr{C}$ of length $2\pi r$. Show (i) that the centre of the circle traces out a curve $\mathscr{D}$ of length $4\pi r$, and (ii) that the region inside $\mathscr{D}$ but outside $\mathscr{C}$ has area $3\pi r^2$.

Question (1960 STEP I Q307)

Sketch the locus (the cycloid) given by

$$x = a(t + \sin t), \quad y = a(1 + \cos t),$$

for values of t between 0 and 4π, where $a > 0$. Find the co-ordinates of the centre of curvature at the point t, and prove that the locus of the centre of curvature is an equal cycloid; illustrate this in your diagram.

Question (1964 STEP II Q104)

A curve is specified by its Cartesian coordinates $x(t)$, $y(t)$. $s(t)$ is the arc-length along the curve, $\psi(t)$ the angle between the tangent to the curve at the point t and the x-axis, $R(t)$ the radius of curvature and $X(t)$, $Y(t)$ are the coordinates of the centre of curvature. Find equations enabling s, ψ, R, X, Y to be calculated. Defining also $S(t)$ as the arc-length along the curve $(X(t), Y(t))$, show that $|dS| = |dR|$, where dS, dR are the infinitesimal changes in S and R corresponding to a change dt in the parameter t. Interpret this result geometrically.

Question (1961 STEP III Q209)

Derive a formula for the area of a surface of revolution. An oblate spheroidal surface is formed by rotation of the ellipse $x^2/a^2 + y^2/b^2 = 1$ about its minor axis. Prove that the area of the surface is $2\pi a^2 \left(1 + \frac{1-e^2}{2e} \log_e \frac{1+e}{1-e}\right)$, where e is the eccentricity $[e = \sqrt{(1 - b^2/a^2)}]$. Given that e is small, find an approximate expression for this area as a sum of powers of e, correct to $O(e^4)$.

Question (1961 STEP III Q210)

A moving point P describes a smooth plane curve, Γ with continuous gradient. The arc AP from a fixed point A on Γ has length s. PP' is the tangent at P in the direction along which s' measures arc the curve from the fixed point A' on Γ, and $PP' = a$ (constant). The locus of P', as P moves, is Γ'. C is the centre of curvature of Γ at P and C' is the centre of curvature of Γ' at P'. Prove $ds'/ds = R/\rho$, where $R = CP$ and $\rho = CP$. Prove also that C' lies on CP' and $CC'/C'P' = -\frac{d\rho d\rho}{R^2 ds}$.

Question (1964 STEP III Q304)

Three particles are simultaneously projected under gravity g in different directions from the same point. Show that after a time t they are at the vertices of a triangle of area proportional to t^2. If the initial velocities of two of the particles are in the same vertical plane and are of magnitudes u, v and elevations α, β respectively, show that the plane of this triangle will pass through the point of projection after a time

$$\frac{2uv\sin(\beta-\alpha)}{g(u\cos\alpha - v\cos\beta)},$$

assuming this to be positive.

Question (1955 STEP II Q109)

Prove that a single loop of the curve $r = 2a\cos n\theta$ $(n > 1)$ has the same area and perimeter as an ellipse with semi-axes a and a/n. If $n = 1 + \epsilon$, where ϵ is small, obtain an approximate expression for the perimeter of the loop as a series of powers of ϵ up to the term containing ϵ^2.

Question (1946 STEP I Q204)

Two triangles $ABC, A'B'C'$ in different planes are so related that AA', BB', CC' meet in a point O. Prove that the lines $BC, B'C'$ meet in a point U, the lines $CA, C'A'$ meet in a point V, the lines $AB, A'B'$ meet in a point W, and that U, V, W are collinear. A variable plane through U, V, W meets OA, OB, OC in A'', B'', C'' respectively. Prove that the vertices of the triangle formed by the lines $A''U, B''V, C''W$ lie on three fixed lines through O.

Question (1946 STEP IIII Q105)

LM and $L'M'$ are lines not in the same plane; N and N' are points on LM and $L'M'$ respectively such that $LM : MN = L'M' : M'N'$. Prove that LL', MM' and NN' are parallel to a plane. L'', M'' and N'' are points on LL', MM' and NN' respectively such that

$$LL' : L'L'' = MM' : M'M'' = NN' : N'N''.$$

Prove that L'', M'' and N'' are collinear.

Question (1948 STEP II Q304)

A circle of radius b rolls round a fixed circle of larger radius a. Find parametric equations for the curve traced out by a point fixed to the circumference of the moving circle. Show that the length of the arc between consecutive cusps is $8(a+b)b/a$ or $8(a-b)b/a$ according as the moving circle is outside or inside the fixed circle. Draw a rough diagram of the curve in each case when $b = a/3$.

Question (1927 STEP I Q108)

O is the centre of a regular polygon of n sides and a is its distance from each side; P is a point (inside the polygon) whose polar co-ordinates referred to O as pole and any initial line are (b, α); $S_m^{(n)}$ is the sum of the mth powers of the distances of P from the sides of the polygon. Shew that

$$S_1^{(n)} = na, \quad S_2^{(n)} = n\left(a^2 + \frac{1}{2}b^2\right).$$

It is a known theorem that $S_3^{(n)}$ is not (like $S_1^{(n)}, S_2^{(n)}$) independent of α; verify this when $n = 3$.

Question (1916 STEP I Q103)

Any two points P, Q are taken on two non-intersecting straight lines, shew that the locus of the middle point of PQ is a plane.

Question (1918 STEP II Q202)

Prove that the eight points of contact of the common tangents of two circles lie upon two straight lines, if the lengths of these common tangents are equal to the sum and difference of the radii of the two circles.

Question (1919 STEP II Q206)

From both ends of a measured base AB the bearings $CAB, CBA, C'AB, C'BA$ of two points C, C' are measured; the four points C, C', A, B lie in a horizontal plane. Find CC' in terms of the measured quantities. If $AB = 2$ miles, $CAB = CBA = 45°$, $C'AB = 30°$ and $C'BA = 60°$, find CC'.

Question (1920 STEP III Q202)

D, E, F are the middle points of the sides BC, CA, AB of a triangle ABC, and points P, Q, R are taken in these sides. Prove that a necessary and sufficient condition that perpendiculars to the respective sides through P, Q, R should be concurrent is

$$BC \cdot DP + CA \cdot EQ + AB \cdot FR = 0.$$

The perpendiculars from the vertices A, B, C of one triangle on the sides $B'C', C'A', A'B'$ of a coplanar triangle are concurrent. Prove that the perpendiculars from A', B', C' on the sides BC, CA, AB respectively are also concurrent.

Question (1920 STEP III Q205)

Prove that the locus of a point in space which is at the same given distance from each of two intersecting straight lines consists of two ellipses with a common minor axis.

Question (1923 STEP III Q202)

Three lines in space do not intersect and are not all parallel to the same plane: prove that they are three edges of a parallelepiped and that one line can be drawn to intersect the three lines so that the intercept on it between a definite pair of the lines is bisected by the third line.

Question (1930 STEP III Q210)

(i) Use homogeneous coordinates to prove that, if two triangles are in perspective, their corresponding sides meet in collinear points. (ii) Shew that the lines joining the vertices of the triangle of reference to the points of intersection of the opposite sides with the conic

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

touch a conic whose equation in tangential coordinates is

$$bcl^2 + cam^2 + abn^2 = 2afmn + 2bgnl + 2chlm.$$

Question (1931 STEP III Q205)

Consider three skew lines, a, b and c, in space. A_1, A_2, A_3 and A_4 are four points on the line a, and l_1, l_2, l_3 and l_4 are the lines through these points which meet b and c. Shew that the cross ratio $(A_1A_2A_3A_4)$ is equal to that of the four planes through a which contain the lines l_1, l_2, l_3 and l_4. (The cross ratio of four planes through a line λ is the cross ratio of the four points in which the planes meet any line which does not meet λ.) By considering homographic ranges on a, or otherwise, shew that, if d is another line, there are two lines which meet a, b, c and d.

Question (1918 STEP IIII Q202)

Prove that it is always possible to draw a straight line to cut two given non-intersecting lines in space at right angles. Prove also that, if the acute angle between the lines is α, then four lines, two lines or none can be drawn to cut the given lines so that the acute angles of the intersections have an assigned value θ, according as the assigned θ is greater than both angles $(\pi - \alpha)/2$ and $\alpha/2$, lies between them or is less than both.

Question (1942 STEP I Q306)

Two planes are inclined at an angle θ. A straight line makes angles α and β with the normals to the two planes. Prove that if its projections on the two planes are perpendicular, then

$$1 - \cos^2\alpha - \cos^2\beta \pm \cos\alpha\cos\beta\cos\theta = 0.$$

Question (1930 STEP II Q303)

Shew that if $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are real quantities satisfying relations

$$l_r^2 + m_r^2 + n_r^2 = 1 \quad (r = 1, 2, 3),$$
$$l_p l_q + m_p m_q + n_p n_q = 0 \quad (\{ {}^p_q \} = 1, 2, 3; p \neq q),$$

then

$$\Sigma l_r^2 = \Sigma m_r^2 = \Sigma n_r^2 = 1; \quad \Sigma l_r m_r = \Sigma m_r n_r = \Sigma n_r l_r = 0,$$

and

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm 1.$$

Question (1938 STEP II Q310)

The line $lx + my + nz = 0$ cuts the sides YZ, ZX, XY of the triangle of reference XYZ in the points L, M, N respectively. L' is the harmonic conjugate of L with respect to Y, Z, and M', N' are similarly defined. O is the point (p, q, r). OL' cuts $M'N'$ at U, and V, W are similarly defined. Prove that XU, YV, ZW are concurrent at the point whose coordinates are given by the equations

$$l(-lp + mq + nr)x = m(lp - mq + nr)y = n(lp + mq - nr)z.$$

Question (1922 STEP I Q405)

Prove that the anharmonic ratio of the pencil formed by joining a variable point on a conic to four fixed points on the conic is constant. Two fixed points A, B are taken on a hyperbola and a variable point P. A line AQR is drawn parallel to one asymptote meeting PB in Q and the parallel through P to the other asymptote in R. The tangent at B and the parallel through B to the latter asymptote meet AR in T and S respectively. Prove that $AT : AS = AQ : AR$.

Question (1914 STEP III Q401)

Two circles A, B cut orthogonally in X and Y. A diameter of A cuts B in P and Q. Prove that the points X, P, Y, Q subtend a harmonic pencil at any point on the circle.

Question (1918 STEP III Q405)

Shew, graphically or otherwise, that the cubic equation in θ,

$$\frac{x^2}{a^2 - \theta} + \frac{y^2}{b^2 - \theta} + \frac{z^2}{c^2 - \theta} = 1, \quad a > b > c,$$

has three real roots λ, μ, ν which are such that $a^2 > \lambda > b^2 > \mu > c^2 > \nu$. Also shew that

$$x^2 = \frac{(a^2 - \lambda)(a^2 - \mu)(a^2 - \nu)}{(a^2 - b^2)(a^2 - c^2)}.$$

Question (1919 STEP III Q406)

The feet of three vertical flagstaffs, of heights α, β, γ, stand at the angular points ABC of a triangle on a horizontal plane. Prove that the inclination to the horizontal of the plane through the tops of the flagstaffs is

$$\tan^{-1}\left[\operatorname{cosec} A\left\{\frac{(\alpha-\beta)^2}{c^2}+\frac{(\alpha-\gamma)^2}{b^2}-\frac{2(\alpha-\beta)(\alpha-\gamma)}{bc}\cos A\right\}^{\frac{1}{2}}\right].$$

Question (1931 STEP II Q503)

A, B, C, D are the vertices of a tetrahedron in which the straight line joining A to the orthocentre of the triangle BCD is perpendicular to the plane BCD. Prove that

$$AB^2+CD^2=AC^2+BD^2=AD^2+BC^2.$$

Determine whether the converse of this theorem is also true.

Question (1925 STEP IIII Q501)

Write a short account of the method of reciprocation showing particularly how to reciprocate a circle into a conic of any species. Apply the method to the following case: S is the focus of a given conic and a line L meets the corresponding directrix in Z. L' is the line joining Z to the pole of L. A second conic having a focus at S touches L, L'. A common tangent to the conic touches them at Q, Q'. Show that QSQ' is a right angle.

Question (1915 STEP I Q606)

The equation of two lines is $ax^2+2hxy+by^2=0$; find the equation of the lines bisecting the angle between them.

If $lx+my=1$ bisects the angle between two lines one of which is $px+qy=1$, shew that the other line is

$$(px+qy-1)(l^2+m^2)=2(pl+qm)(lx+my-1).$$

Question (1923 STEP I Q606)

Find the equation of the bisectors of the angles between the lines

$$ax^2+2hxy+by^2=0.$$

The x-axis is reflected in each line of the pair $ax^2+2hxy+by^2=0$. Prove that the equation of the reflexions is

$$4abx^2+4h(a-b)xy+\{(a+b)^2-4h^2\}y^2=0.$$

Question (1926 STEP III Q607)

Through a point P two lines are drawn in given directions. Prove that, if the line joining the middle points of the intercepts made by these lines on two fixed perpendicular lines OA, OB is perpendicular to the line OP, the locus of P is either of two fixed perpendicular lines.

Question (1925 STEP I Q710)

Explain how to distinguish the two "sides" of a bilateral surface. Define $\iint f(x,y,z)dydz$ taken over a specified side of a given bilateral surface and show how to calculate it in terms of u and v when the equations to the surface are

$$x = \theta(u,v), \quad y = \phi(u,v), \quad z = \psi(u,v).$$

Find the value of

$$\iint x^3y^3z^5dydz$$

taken over the outer side of the octant of the surface of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

for which x, y, z are all positive.

Question (1920 STEP III Q707)

Define the terms: vector product, scalar field, vector field, gradient, divergence, curl, indicating which of the three latter apply to scalars and which to vectors. Prove that the necessary and sufficient conditions that a field of force $\mathbf{F}$ be derivable (a) from a unique scalar potential (i.e. as its gradient), (b) from a unique vector potential (i.e. as its curl) are respectively: curl $\mathbf{F} = 0$, div $\mathbf{F} = 0$ throughout the space considered. Find expressions for the scalar and vector potentials of a magnetic particle of moment I.

Question (1922 STEP I Q807)

A developable surface is commonly defined

(a) as the envelope of a plane whose equation contains one parameter,

(b) as the surface generated by the tangents to a twisted curve.

Investigate the equivalence of these definitions.

Question (1923 STEP I Q801)

Show how the self-corresponding points of two co-basal homographic ranges may be determined. Give a construction for the points, if any, in which a conic through five given points meets a given straight line.

Question (1923 STEP I Q804)

Define the curvature (κ) and torsion (τ) of a twisted curve, explaining carefully any conventions of sign that you make, and prove from your definition that, with the usual notation,

$$s'^3\kappa^2\tau = \pm \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix},$$

determining by means of your conventions which sign is to be taken.

Question (1924 STEP II Q810)

The magnetic vector-potential $\mathbf{U}$ in a magnetic field $\mathbf{H}$ is defined to be any vector function satisfying the relation

$$\mathbf{H} = \text{curl } \mathbf{U}.$$

Show that the field at a point distant r from a doublet at (x, y, z) of strength represented by the vector $\boldsymbol{\mu}$ may be derived from a vector-potential given by

$$\mathbf{U} = \boldsymbol{\mu} \wedge \text{grad}\left(\frac{1}{r}\right),$$

and hence that the vector-potential due to a normally magnetised magnetic shell of uniform strength ϕ may be taken to be

$$\mathbf{U} = \phi \oint \frac{1}{r} d\mathbf{s},$$

where $d\mathbf{s}$ is an element of arc of the boundary curve of the shell and the integral is taken round the boundary curve. Deduce that the mutual potential energy of two currents of intensities i, i' in closed circuits may be expressed in the form

$$-ii' \int\int \frac{1}{r} d\mathbf{s}.d\mathbf{s}',$$

the integrals being taken round the circuits.