Question (1970 STEP I Q2)

Find a relationship between x, y and z which must hold if there are to exist p, q and r such that x, y and z are respectively the pth, qth and rth terms both of an arithmetical and of a geometrical progression.

Question (1974 STEP I Q4)

For a positive integer N, $\sigma(N)$ denotes the sum of all the positive integers which divide N (including 1 and N). If $N = p^n$ where p is prime, show that

$$\sigma(N) = \frac{p^{n+1} - 1}{p - 1}.$$

Show further that, for an arbitrary positive integer N which factorizes as $p_1^{n_1} p_2^{n_2} \dots p_s^{n_s}$, where $p_1, \dots, p_s$ are distinct primes,

$$\sigma(N) = \sigma(p_1^{n_1}) \dots \sigma(p_s^{n_s}).$$

Deduce that if N is an odd integer such that $\sigma(N)$ is also odd, then N is a square.

Question (1976 STEP I Q2)

Show that the sum of the first n odd positive integers is a perfect square. The odd positive integers are arranged in blocks with n integers in the nth block and a_n denotes the sum of the numbers in the nth block, so that $a_1 = 1$, $a_2 = 3 + 5$, $a_3 = 7 + 9 + 11$, etc. Find an expression for a_n and deduce that $\displaystyle\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n+1)^2$.

Show that $\displaystyle\sum_{r=1}^{n} r^5 = \frac{1}{12} n^2 (n+1)^2 (2n^2 + 2n - 1)$ by considering the situation in which the nth block has n^2 integers.

None

Question (1977 STEP I Q4)

Let x be any real number. The symbol $[x]$ denotes the greatest integer less than or equal to x (e.g. $[-4\frac{1}{2}] = -5$, $[\pi] = 3$). Which of the following statements are true and which false? Prove the true ones, and, for each of the false ones, give an example in which the statement fails to hold. (i) $[x + y] \geq [x] + [y]$. (ii) $[x^2] = [x]^2$. (iii) $[[x]/n] = [x/n]$ for every positive integer n. (iv) $[nx] = n[x]$ for every positive integer n. Show that

$$\sum_{k=0}^{n-1} \left[a + \frac{k}{n} \right] = [na]$$

for any real number a and any integer $n \geq 1$. [Hint: deal first with the case $0 \leq a < 1/n$ and then consider the effect of replacing a by $a + 1/n$ and $a - 1/n$.]

Question (1984 STEP II Q1)

If $0 \geq a_i \geq -1$ for all i show that $\prod_{r=1}^{n}(1+a_r) \geq 1 + \sum_{r=1}^{n} a_r$. Show that the inequality is also true if $a_i \geq 0$ for all i. Is it true if $a_i > -1$ for all i?

Question (1974 STEP IIII Q9)

S is a set of n points P_1, P_2, ..., P_n equally spaced round the periphery of a circle (i.e. at the vertices of a regular polygon). A is a subset of them, a in number; B is another, b in number. χ_A, the 'characteristic function of A' is a function defined only on the points of S, thus: $\chi_A(P_i) = 1$ if P_i belongs to A, $\chi_A(P_i) = 0$ otherwise; χ_B is defined similarly. B_k is the set obtained by rotating B through an angle $2k\pi/n$ positively about the centre of the circle (so that B_k is also a subset of S, and is congruent to B). By considering the double sum

$$\sum_{k=1}^{n}\sum_{i=1}^{n}\chi_A(P_i)\chi_{B_k}(P_i),$$

show that there exists a value of k for which $A \cap B_k$ contains at least ab/n points.

Question (1964 STEP IIII Q206)

Suppose that a_j, b_j $(1 \leq j \leq n)$ are given real numbers and that

$$1 \leq a_j \leq A, \quad 1 \leq b_j \leq B \quad (1 \leq j \leq n)$$

for some A, B. Show that

$$a_j b_j \geq u_j B + v_j A \quad (1 \leq j \leq n),$$

where u_j, v_j are defined by the equations

$$a_j^2 = u_j + v_j A^2, \quad b_j^2 = u_j B^2 + v_j.$$

Deduce that

$$\frac{(\sum a_j^2)(\sum b_j^2)}{(\sum a_j b_j)^2} \leq \left(\frac{(AB)^{\frac{1}{2}} + (AB)^{-\frac{1}{2}}}{2}\right)^2.$$

Question (1919 STEP I Q101)

Equal weights are suspended from the joints of a chain composed of five straight light smoothly jointed links. The extreme links are fastened to two points P, Q in the same horizontal line by smooth joints. The projection of each link on the horizontal is equal to a, the distance PQ being $5a$. The depth of the lowest link, which is horizontal, below PQ is $6a$. Find the inclination of each link to the horizontal.

Question (1915 STEP IIII Q206)

Find the sum of 18 terms of the series $10 + 8\frac{3}{5} + 7\frac{1}{5} + \dots$.

Find also what term of the series is equal to $-\frac{1}{5}$.

Question (1917 STEP IIII Q207)

Find the sum of 24 terms of the series $4\frac{1}{2} + 3\frac{3}{4} + 3 + \dots$. The sum of eight terms of an arithmetical progression is 24 and its fifth term is 1; find the first term and the common difference.

Question (1918 STEP IIII Q207)

In an Arithmetic Progression the 9th term is 7 times the 1st term and the sum of the 4th and 6th terms is 32; find the series.

Question (1921 STEP III Q309)

A set of numbers $a_1, a_2, a_3, \dots, a_n, \dots$, is such that from the third onwards each is the arithmetic mean between its two immediate predecessors: prove that

$$a_n = A + B(-\frac{1}{2})^n,$$

where A and B are independent of n and are to be found in terms of a_1 and a_2.

Question (1913 STEP IIII Q612)

Find a formula for the nth term of an A.P. whose first term is a and common difference d, and determine the sum of the series $1 + 3 + 5 + 7 + \dots$ to n terms.