**Question** (1967 STEP I Q6)

The vertices $A_1, A_2, A_3, A_4, A_5$ of a regular pentagon lie on a circle of unit radius with centre at the point $O$. $A_1$ is the mid-point of $OP$. Prove that

(i) $PA_1 \cdot PA_2 \cdot PA_3 \cdot PA_4 \cdot PA_5 = 31$;

(ii) $\sum_{r=1}^{5} (PA_r)^2 = 25$;

(iii) $A_1A_2 \cdot A_1A_3 \cdot A_1A_4 \cdot A_1A_5 = 5$.

None

**Question** (1968 STEP I Q9)

$P$ and $Q$ are points of the plane outside the circumcircle of the regular polygon $A_0A_1A_2 \ldots A_{n-1}$ whose centre is the point $O$. The line-segment $OP$ contains a vertex of the polygon, while the segment $OQ$ perpendicularly bisects an edge. By representing this situation in the complex plane, or otherwise, show that the geometric mean of all the lengths $QA_r$ exceeds the length $QO$, while the length $PO$ exceeds the geometric mean of all the lengths $PA_r$.

None

**Question** (1963 STEP I Q105)

Points $A_1$, $A_2$, $\ldots$, $A_n$ (where $n \geq 3$) are equally spaced round the circumference of a circle. Their distances from a line drawn through the centre are $d_1$, $d_2$, $\ldots$, $d_n$. Prove that

$$d_1^2 + d_2^2 + \ldots + d_n^2$$

is the same for every direction of the line.

None

**Question** (1961 STEP I Q410)

A closed polygon of $2n$ sides, $n$ of which are of length $a$ and $n$ of length $b$, is inscribed in a circle. Show that the radius of the circle is independent of the arrangement of the sides, and find its value.

None

**Question** (1963 STEP IIII Q106)

By considering the sum of the roots of the equation $z^5 = 1$, find an equation with integer coefficients which is satisfied by $\cos \frac{2\pi}{5}$, and hence obtain an expression for $\cos \frac{2\pi}{5}$. Prove the theorem (known to Euclid) that if a pentagon, a hexagon, and a decagon, regular and with sides $a_5$, $a_6$, $a_{10}$ are inscribed in the same circle, then

$$a_5^2 = a_6^2 + a_{10}^2.$$

None

**Question** (1964 STEP II Q202)

Two regular polygons of $n_1$ and $n_2$ sides are inscribed in two concentric circles of radii $r_1$ and $r_2$ respectively. Prove that the sum of the squares on all the lines joining the vertices of one to the vertices of the other is

$$n_1 n_2 (r_1^2 + r_2^2).$$

None

**Question** (1953 STEP II Q303)

Let

$$\rho = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m},$$

where $m$ is a positive integer. For any integer $r$ put

$$p_m(r) = \frac{\rho^r}{1-\rho} + \frac{\rho^{2r}}{1-\rho^2} + \cdots + \frac{\rho^{(m-1)r}}{1-\rho^{m-1}}.$$

By considering the differences

$$p_m(r+1) - p_m(r)$$

and the sum

$$p_m(0) + p_m(1) + \cdots + p_m(m-1),$$

or otherwise, evaluate the $p_m(r)$ for all $m$ and $r$. Show in particular that

$$p_m(0) = \frac{1}{2}(m-1).$$

**Question** (1944 STEP I Q105)

Prove that

$$\sin 3\theta = 4\sin\theta \sin(\theta + \tfrac{1}{3}\pi)\sin(\theta + \tfrac{2}{3}\pi).$$

The trisectors of the angles of a triangle ABC meet in $X, Y, Z$ ($X$ being the point of intersection of the trisectors of B and C lying nearest to BC, and similarly for $Y$ and $Z$). Express the ratio $AY/AZ$ as simply as you can in terms of the angles of the triangle ABC, and hence find the angles of the triangle AYZ. Hence, or otherwise, prove that XYZ is an equilateral triangle.

**Question** (1915 STEP I Q108)

Prove that

$$\sum_{r=0}^{n-1} \frac{1}{1 - \cos\left(\phi + \frac{2r\pi}{n}\right)} = \frac{n^2}{1 - \cos n\phi}.$$

**Question** (1923 STEP I Q111)

Prove that
$$2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\dots\sin\frac{(n-1)\pi}{n}=n.$$
Hence or otherwise prove that $\displaystyle\int_0^{\pi}\log\sin x dx=-\pi\log 2.$

**Question** (1919 STEP I Q105)

Four real or complex numbers (other than zero) are such that their squares are the same numbers in the same or a different order; prove that each number is a root of unity.

**Question** (1916 STEP II Q204)

Prove that for the continued fraction $a_1+\frac{1}{a_2+}\frac{1}{a_3+}\dots$ where the $a$'s are all positive, any convergent is intermediate in magnitude between the next two preceding ones. For the fraction $a+\frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\dots$, prove that if $p_n/q_n$ is the $n$th convergent, $p_{2n}=q_{2n+1}$ and $bp_{2n-1}=aq_{2n}$.

**Question** (1917 STEP II Q207)

Shew that the problem of determining the $n$th roots of 1 is equivalent to that of inscribing a regular polygon of $n$ sides in a circle. If $n$ denote an even integer, shew that the product
$$\left(x^2+\cot^2\frac{\pi}{2n}\right)\left(x^2+\cot^2\frac{3\pi}{2n}\right)\left(x^2+\cot^2\frac{5\pi}{2n}\right)\dots\left(x^2+\cot^2\frac{(n-1)\pi}{2n}\right)$$
is equal to $\frac{1}{2}\{(1+x)^n+(1-x)^n\}$.

**Question** (1925 STEP II Q206)

Find the real linear and quadratic factors of $z^n-1$ when $n$ is an odd positive integer. Deduce that
$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\dots\sin\frac{(n-1)\pi}{n}=\frac{n}{2^{n-1}}.$$

**Question** (1920 STEP III Q210)

Prove that, if $r$ is prime to $n$ and $\alpha=\cos\frac{2r\pi}{n}+i\sin\frac{2r\pi}{n}$, the $n$th roots of unity are $1,\alpha,\alpha^2,\dots,\alpha^{n-1}$. Shew that, if $p$ is prime to $n$,
$$1+\alpha^p+\alpha^{2p}+\dots+\alpha^{(n-1)p}=0.$$

**Question** (1930 STEP IIII Q204)

If $P_0, P_1, \ldots, P_{n-1}$ are $n$ equidistant points round a circle of unit radius, and $a_r$ is the distance $P_0P_r$, prove that $a_1a_2\ldots a_{n-1} = n$. Find also $a_1+a_2+\cdots+a_{n-1}$ and deduce that when $n$ is large the average distance of the points from $P_0$ is approximately $4/\pi$.

**Question** (1924 STEP II Q304)

If $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ are the fifth roots of unity, prove that

$$\alpha\tan^{-1}\alpha + \alpha^2\tan^{-1}\alpha^2 + \alpha^3\tan^{-1}\alpha^3 + \alpha^4\tan^{-1}\alpha^4$$

$$= \pi\cos\frac{3\pi}{5} + \sin\frac{3\pi}{5}\log\left(\tan\frac{\pi}{20}\right) + \sin\frac{\pi}{5}\log\left(\tan\frac{3\pi}{20}\right).$$

**Question** (1941 STEP III Q305)

If $x$ is any complex root of the equation $x^{11} - 1 = 0$, and if

$$a = x + x^3 + x^4 + x^5 + x^9, \quad b = x^2 + x^6 + x^7 + x^8 + x^{10},$$

prove that $(a-b)^2 = -11$. Show further that

$$(x^3+1)[a - b - 2(x - x^{10})] = x^3 - 1,$$

and deduce that

$$\tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} = \sqrt{11}.$$

**Question** (1926 STEP I Q410)

If $x$ and $\theta$ are real, and $n$ is a positive integer, express $x^{2n} - 2x^n\cos n\theta + 1$ as the product of $n$ real quadratic factors.

**Question** (1916 STEP II Q404)

Prove that, in a triangle $ABC$,

$$\Sigma\sin^2 A\tan A = \tan A\tan B\tan C - 2\sin A\sin B\sin C.$$

**Question** (1939 STEP II Q409)

Express

$$\begin{vmatrix} \sin^3\theta & \sin\theta & \cos\theta \\ \sin^3\alpha & \sin\alpha & \cos\alpha \\ \sin^3\beta & \sin\beta & \cos\beta \end{vmatrix}$$

as the product of four sines and hence find all values of $\theta$, in terms of $\alpha$ and $\beta$, for which the value of this determinant is zero.

**Question** (1942 STEP II Q410)

By considering the expression for $\cos 7\theta$ in terms of $\cos\theta$, find the roots expressed in trigonometric form of the equation

$$64x^6 - 112x^4 + 56x^2 - 7 = 0.$$

**Question** (1923 STEP II Q506)

Shew how to determine the four fourth roots of a complex expression of the form $a + ib$.

**Question** (1932 STEP III Q505)

If $\omega$ is one of the imaginary $n$th roots of unity, shew that

$$\sum_{r=1}^{n-1} \frac{1-\omega^r}{y-\omega^r} = \frac{n(y^{n-1}-1)}{y^n - 1}.$$

By the use of the calculus, or otherwise, prove that if $x > 1$, then

$$(n+1)^2(x+3)(x-1) > 4n^2\{x^{n+1}+(n+1)x^n-n-2\}-4n(n+1)\sum_{r=1}^{n-1}(1-\omega^r)\log\frac{1}{1-\omega^r x^{-1}}$$

$$> 4(n+1)^2(x-1),$$

where $n-1$ is a positive integer, and $x^n$ is real.