Question (1971 STEP II Q10)

Two smooth planes meet at right angles in a horizontal line. A rod, whose density is not necessarily uniform, is placed above this line and perpendicular to it, and rests on the planes. If the steeper plane is inclined at an angle θ to the horizontal, find the equilibrium positions of the rod. Discuss explicitly the following special cases:

(i) the density of the rod is uniform,

(ii) the density of the rod is proportional to the distance from one end.

Question (1970 STEP III Q15)

Five equal uniform bars, each of mass M, are freely jointed together to form a plane pentagon $ABCDE$. They are suspended from A, and are constrained by equal light strings AC and AD so as to form a regular pentagon. Show without direct calculation that the tension in each string is the same as it would be if the bars were replaced by light rods and a mass M attached at each vertex. Hence show that this tension has magnitude $2Mg\cos\frac{1}{5}\pi$.

Question (1974 STEP III Q14)

A four-wheeled truck runs forward freely on level ground. The distance between the front and rear axles is D, and the centre of gravity of the truck is at a distance β from the vertical plane through the front axle and at a height H above the ground. The moments of inertia of wheels and axles are negligible. Find the deceleration of the truck if the rear wheels become locked (the front wheels remaining free), and μ is the coefficient of friction between the wheels and the ground. If the front rather than the rear wheels become locked show that the rear wheels remain on the ground provided that $\mu < \beta/H$.

Question (1972 STEP IIII Q15)

A tumbler which has square cross-section of side $2a$ and height Ka is closed at one end and this end rests on a rough horizontal table. The tumbler is filled to a height ka with liquid of uniform density. Assuming that no sliding takes place and that the weight of the tumbler is negligible compared with that of the liquid, show that when the table is tilted slowly through an angle θ about an axis parallel to one face of the tumbler then, provided

$$\tan\theta \le k \le K - \tan\theta,$$

the tumbler will topple when θ is given by

$$\tan^3\theta + (3k^2+2)\tan\theta - 6k = 0.$$

If it is required to tilt the table through an angle $\tan^{-1}\frac{1}{2}$ without spillage, determine what height of tumbler is required and how full it can be.

Question (1960 STEP IIII Q107)

A pile of n bricks is in equilibrium, each brick resting horizontally on the one and their long sides lying in the same vertical north-south planes. The bricks are uniform rectangular blocks of the same material, of length a and height b. The sun is due south at an elevation α. Find the minimum length of the shadow of the pile (in the north-south direction) in the following two cases:

(i) $\frac{a \tan \alpha}{b} > n$;

(ii) $\frac{a \tan \alpha}{b} < 2$.

[By definition the shadow includes the area under the bottom brick. The sun may be assumed to be a point source.]

Question (1960 STEP III Q101)

A uniform rigid rod AB of length 5 inches and weight w hangs from a point O by two inextensible strings AO, BO of lengths respectively 3 and 4 inches. A variable weight W is attached at B. Find the tension in OA, and verify that it decreases as W increases.

Question (1961 STEP III Q101)

A uniform rod AB is suspended from a point O by light inelastic strings OA, OB attached to its ends. Prove that the tensions in the strings are proportional to their lengths. Examine whether the result can be extended to (i) a uniform triangular lamina ABC suspended by strings OA, OB, OC; (ii) a uniform polygonal lamina with four or more vertices suspended by strings attached to its vertices.

Question (1963 STEP III Q101)

When it is on level ground, the centre of gravity of a motor car is at height h and its front and rear axles are at horizontal distances a and b from the centre of gravity. If it is parked facing up a slope which makes an angle α with the horizontal, with its rear wheels locked, show that the coefficient of friction between the rear wheels and the road must not be less than $(a + b)/(h + a \cos \alpha)$. What is the corresponding result if it is parked facing down the slope?

Question (1958 STEP III Q201)

A pedestal is constructed of three uniform right circular cylinders placed with their axes vertical and in the same line. The weights of the cylinders are in the ratios 2:1:3, and their radii are in the ratios 12:11:9, where the cylinders are taken in order from the topmost downwards. If no mortar is used in the pedestal, find the greatest weight of a statue which may be placed safely anywhere on the top of the pedestal, in terms of the weight of the middle cylinder. If the three cylinders are cemented together, while the base is still not fixed to the ground, show that the greatest weight of a statue which may now be placed safely on the top of the pedestal is $1 + k$ times its value when the cylinders were not cemented.

Question (1958 STEP III Q202)

A light rigid wire is bent into the shape of a rectangle $ABCD$, with $AB = a$, $BC = b$. Particles of weights w, $5w$, w, $2w$ are attached to the vertices, A, B, C, D respectively, and the wire is then suspended freely from A. What is the inclination of AB to the vertical when the system composed of wire and particles is in equilibrium? A rough horizontal plane is held so as to touch the wire at C, and another particle of weight w is attached to C. If μ is the coefficient of friction between the wire at C and the plane, what further force must be applied vertically upwards to the plane so that sliding will just commence at C?

Question (1961 STEP III Q202)

Calculate the position of the centroid of a uniform hemisphere. A solid is shaped by cutting out from a uniform hemisphere of radius R a sphere of radius $\frac{1}{2}R$. This solid rests with its plane face in contact with a rough inclined plane, and a gradually increasing force is applied at the pole of the hemisphere in the direction parallel to the line of greatest (upward) slope of the plane. The coefficient of friction is μ and the angle of inclination of the plane is α. Show that the solid slides or tilts first according as

$$\mu \lessgtr 1 - \frac{3}{2}\tan\alpha.$$

Question (1958 STEP III Q301)

A particle of weight $2W$ is attached to the end A, and a particle of weight W attached to the end B, of a light rod AB of length $2a$. The rod hangs from a point O by light strings AO, BO, each of length b. Prove that in equilibrium the inclination of the rod to the horizontal is θ, where

$$\tan\theta = \frac{a}{3\sqrt{(b^2 - a^2)}}.$$

Find the tension in the string AO in terms of a, b, and W.

Question (1962 STEP III Q310)

A uniform rod of length $2a$ is supported symmetrically in a horizontal plane by two pegs distant $2d$ apart. A second uniform rod of equal length but of mass twice that of the first is suspended by two equal inextensible vertical cords from the ends of the first rod. If one cord is cut find the smallest value of d for which the upper rod will not begin to tilt.

Question (1963 STEP III Q301)

A, B and C are three smooth horizontal parallel pegs, A and C being a distance a from B in a horizontal plane. Two uniform rods, DB and EF, of the same weight per unit length and of length $3a$ and a, respectively, are smoothly jointed at E and are laid perpendicularly across the pegs A, B, C. Show that, if the rods are to remain horizontal, D must be $\frac{5a}{6}$ beyond an end peg.

Question (1958 STEP III Q404)

A flat plate of uniform thin material is in the form of a plane quadrilateral $ABCD$. The diagonals meet at a point O. Show that its centre of mass coincides with that of four particles each of mass m at A, B, C, D and one of mass $-m$ at O.

Question (1960 STEP III Q401)

The ends A, B of a light rod AB are joined by light inextensible strings AO, BO to a fixed point O, and AO and BO are equal in length and perpendicular to each other. If weights W_1 and W_2 are now suspended from A and B, find the angle to the horizontal that the rod will take up in equilibrium.

Question (1950 STEP II Q209)

A long plank of length $2l$ and mass m is supported horizontally at its two ends by vertical ropes, the weaker of which can only stand a tension $\frac{5}{4}mg$. A man of mass m walks across the plank starting at the stronger rope. When the weaker rope breaks the man clings to the plank at the position he has reached. Show that when the weaker rope breaks the tension in the stronger rope suddenly becomes $1\frac{3}{16}mg$.

Question (1952 STEP II Q209)

The centre of mass of a car, moving in a straight line on level ground, is at height h above ground level and at a distance a from the vertical plane through the rear axle and b from the vertical plane through the front axles. Show that if braking is applied equally to the two rear wheels only, excessive braking may cause a skid but cannot cause the wheels to leave the ground; but that if braking is applied to the front wheels the rear wheels may leave the ground if the coefficient of friction μ is great enough; and find the condition for this to happen, neglecting the rotatory inertia of the revolving parts. If the braking force is divided between the front and back wheels, determine whether it is possible to get more effective braking than with front-wheel brakes only, and whether the result is any different if the car is running downhill. If the angular momentum of all rotating parts may be assumed to be in the same direction as that of the wheels, and proportional to the car's speed, determine whether the maximum attainable retardation is greater or less than if this angular momentum were negligible.

Question (1953 STEP II Q207)

A number n of equal uniform rectangular blocks are built into the form of a stairway, each block projecting the same distance a beyond the one below. The top block is supported from below at its outer edge. Show that the stairway can stand in equilibrium if, and only if, $2l > a(n-1)$, where $2l$ is the width of each block.

Question (1953 STEP II Q210)

A uniform solid cube of side $2a$ starts from rest and slides down a smooth plane inclined at an angle $2\tan^{-1}\frac{1}{4}$ to the horizontal, the orientation of the cube being such that its front face is perpendicular to the lines of greatest slope of the plane. The cube meets a fixed horizontal bar placed perpendicular to the direction of motion and at a perpendicular distance $a/4$ from the plane. Show that, if the cube is to have sufficient velocity to surmount the obstacle when it reaches it, the cube must be allowed first to slide down the plane through a distance greater than $107a/60$. (The obstacle may be taken to be inelastic and so rough that the cube does not slip on it.)

Question (1950 STEP II Q307)

The ends A, B of a heavy uniform rod of weight w and length $2a$ are attached by two light inextensible strings each of length b to two points C, D at the same level a distance $2c$ apart where $a + c > b > \sqrt{(a^2 + c^2)}$. The rod is now moved in such a way that it always remains horizontal and so that the mid-point of AB remains vertically below the mid-point of CD. Find the potential energy of the rod as a function of the angle between AB and CD, if both strings remain taut and do not cross. Also find the couple necessary to keep AB perpendicular to CD.

Question (1953 STEP II Q306)

A rectangular picture frame hangs from a smooth peg by a string of length $2a$ whose ends are attached to two points on the upper edge at distances c from its middle point. Prove that if the depth of the frame exceeds $2c^2(a^2 - c^2)^{-1/2}$ there is no position of equilibrium except that in which the picture frame hangs symmetrically.

Question (1955 STEP II Q304)

The uniform scalene triangular lamina ABC is at rest in equilibrium freely suspended from a point K by three equal light inextensible strings KA, KB, KC. Prove that the Euler line of the triangle ABC is a line of greatest slope of the plane ABC. [The Euler line is the line containing the circumcentre, centroid, nine-point centre and orthocentre.]

Question (1956 STEP II Q310)

A uniform cylinder, whose normal cross-section is an ellipse with eccentricity e, is placed with its generators horizontal on a perfectly rough plane inclined at an angle α to the horizontal. Show that, if it is released from rest, complete revolutions will occur, whatever the initial position, provided that

$$\tan\alpha > \frac{e^2}{2\sqrt{(1-e^2)}}.$$

Question (1950 STEP III Q107)

Define the mass-centre of n coplanar point-masses m_i $(i = 1, 2, \ldots, n)$, situated at points (x_i, y_i), and prove that it is a unique point. If the coordinates x_r, y_r of one of the masses m_r are changed to $(x_r + \xi_r, y_r + \eta_r)$, show that the coordinates of the mass-centre are changed by $(m_r\xi_r/M, m_r\eta_r/M)$, where M is the total mass. Generalise this to cover the case in which any number of the point-masses are moved in their plane. A circular disc of uniform thin paper, of radius a, is cut along a radius and one of the quadrants is folded over so as to lie in the plane of the remaining three quadrants. Find the distance of the mass-centre of the folded paper from the centre of the circle. [It may be assumed that the mass-centre of a sector of the circle, of angle 2β, is at a distance $(2a\sin\beta)/(3\beta)$ from the centre of the circle.]

Question (1951 STEP III Q102)

A straight rod ABC of weight $3W$ rests horizontally on a nearly flat surface, making contact only at the points A, B and C, where B is the mid-point of AC. The normal reaction at each of the points of contact is W and the coefficient of friction is μ. A gradually increasing horizontal force at right angles to the rod is applied to the rod at A. Find the greatest magnitude of this force for which equilibrium is possible, and describe how the equilibrium is broken.

Question (1952 STEP III Q103)

A uniform rigid square lamina $ABCD$, of weight W, rests, with the diagonal AC vertical and A uppermost, on two parallel horizontal rails which are perpendicular to the plane of the lamina and are in contact with the lamina at E, F, the mid-points of DC, CB respectively. It may be supposed that the plane of the lamina is kept vertical by smooth constraints. A force P, parallel to DB, is applied at A and is increased from zero, and equilibrium is ultimately broken. If this occurs by the lamina beginning to rotate about F, show that the coefficient of friction at F is at least $\frac{1}{2}$. If, on the other hand, equilibrium is broken by slipping at E and F, where the coefficients of friction are μ_1, μ_2 respectively, show that, when slipping occurs, the ratio of the normal reaction at E to that at F is $(1 - 2\mu_2) : (1 + 2\mu_1)$. Show also that in these circumstances the value of P is then

$$\frac{W(\mu_1 + \mu_2)}{2 + \mu_1 - \mu_2 + 4\mu_1\mu_2}.$$

Question (1954 STEP III Q102)

Two equal uniform cubes, each of weight W, stand on a horizontal table with a small gap between them, and with the line joining their centres perpendicular to adjacent vertical faces. A light wedge of semi-angle $\beta(< \frac{1}{4}\pi)$ lies symmetrically, vertex downwards, between the cubes and is acted on by a slowly increasing vertical force P until equilibrium is broken. The coefficient of friction between either cube and the table is $\mu(< \cot\beta)$, and there is no friction between the wedge and either cube. Find the values of P for which equilibrium would be broken on the assumption (a) that the cubes slide, (b) that they topple outwards. Hence show that the equilibrium is broken by the cubes sliding if $\mu < 1/(2 - \tan\beta)$.

Question (1954 STEP III Q103)

Show that the centre of mass of a sector, of angle 2α, cut from a uniform thin circular disc of radius a, is distant $(2a\sin\alpha)/3\alpha$ from the centre of the circle. A radial cut OA is made in a uniform thin circular disc, of centre O and radius a. A quadrantal portion AOB is folded over so that AOB lies in contact with the other portion of the disc. Neglecting the thickness of the disc, find the distance from O of the centre of mass of the folded disc.

Question (1955 STEP III Q106)

A thin uniform rod rests at one end on a horizontal plane while the other end is slowly raised by means of a wedge forced under it, the axis of the rod being coplanar with a line of greatest slope of the wedge. The angle of the wedge is $45°$ and the coefficient of friction at the points of contact of the rod with the plane and with the wedge is $\frac{1}{2}$. Show that until the rod reaches an angle $\tan^{-1}\frac{1}{2}$ to the horizontal an additional force will be required at the lower end to prevent it slipping.

Question (1950 STEP III Q201)

Two uniform planks each of length l and weight W are freely hinged to the ground at two points distant kl apart. A third plank of length l and weight W' rests horizontally at its points of trisection on the other two planks when these are equally inclined upwards and towards each other. The coefficient of friction between the planks is μ. Find an equation giving the greatest possible value of k and show that this value increases with W'.

Question (1951 STEP III Q201)

A heavy uniform rod of length $2l$ is placed in a vertical plane so that it is partly supported by a rough horizontal peg while its lower end rests against a smooth vertical wall. The axis of the peg is parallel to the wall and is at a distance d from it, and the co-efficient of friction between the peg and the rod is μ. The rod is inclined at an angle α to the wall and $l\sin\alpha > d$. Show that the rod will not slip, if

$$l\sin^3\alpha(1+\mu\cot\alpha) \geq d, \quad \text{or} \quad l\sin^3\alpha(1-\mu\cot\alpha) \leq d$$

according as

$$l\sin^3\alpha < d, \quad \text{or} \quad l\sin^3\alpha > d$$

respectively.

Question (1951 STEP III Q205)

A thin uniform heavy rod AB is bent into a semicircle of radius a, and is hung by a light inextensible string of length l which is attached to the ends of the rod and passes over a smooth peg. Derive an equation which gives the values of the inclination of the diameter AB to the horizontal for which equilibrium is possible.

None

Question (1952 STEP III Q203)

A quadrilateral $ABCD$ is formed from four uniform rods freely jointed at their ends. The rods AB and AD are equal in length and weight, and so also are the rods BC and CD. The quadrilateral is suspended from A and a string joins A and C so that ABC is a right angle and $BAD = 2\theta$. Show that the tension in the string is $w' + (w + w')\sin^2\theta$, where w is the weight of AB and w' is the weight of BC.

Question (1954 STEP III Q203)

Two uniform rods AB, BC, equal in weight and length, are freely jointed together at B, and stand in a vertical plane with the ends A and C on a rough horizontal plane, each rod being inclined at an angle θ to the horizontal. The coefficient of friction at A and C is μ. Show that, for equilibrium, $\mu \geq \cot\theta$. A gradually increasing force is applied at B in a direction parallel to AC. Determine how the way in which equilibrium is broken depends on the values of μ and θ.

Question (1950 STEP III Q301)

Prove that a coplanar system of forces may be reduced to a force through an assigned point and a couple. Show also that, in general, the system is equivalent to a single force. A square lamina $ABCD$ lies on a smooth horizontal table and is subject to a force F acting at A along DA produced, a force $2F$ at B along AB produced, and a force $2F$ at C towards B. Show that if the lamina is freely hinged at D it will not move. Find the force on the hinge.

Question (1950 STEP III Q303)

A closed rectangular box is made of thin uniform sheet, its base being a square of side a and its height $\frac{3}{2}a$. The base is made of double thickness of sheet and the rest of the box is made of sheet of single thickness. The box stands on a perfectly rough inclined plane four of its edges being parallel to the line of greatest slope. A horizontal force equal to one-third of the weight of the box is applied along the perpendicular bisector of the highest edge so as to tend to topple the box down the slope. Show that if the force is just able to topple the box then the inclination of the plane to the horizontal is $\tan^{-1}\frac{5}{8}$.

Question (1950 STEP III Q304)

A plane uniform lamina is bounded by a semicircle of radius a. Find its centre of gravity. A second plane uniform lamina, of different surface density, is bounded by a square of side $2a$. A composite plane lamina is formed by joining the base of the semicircular lamina to a side of the square one. What is the ratio of the surface densities if the centre of gravity of the composite lamina is at the mid-point of the common edge?

Question (1951 STEP III Q302)

A tripod formed of three uniform rods OA, OB, OC, which are of the same weight and of the same length $2a$ and are freely jointed together at O, rests on a rough horizontal plane so that the feet A, B, C form an equilateral triangle. If the coefficient of friction is $\frac{1}{4}$, prove that the least possible height of O above the plane is approximately $1.79a$.

None

Question (1952 STEP III Q302)

A square table of weight W has side $2a$ and height b. The top is uniform and it has four equal legs at its corners. It stands on a rough horizontal floor of coefficient of friction μ. Given that

$$\frac{b}{a} < \frac{1-\mu^2}{\mu},$$

find the magnitude and elevation of the least force that will make the table slide on the floor parallel to a side. Show also that this force may be applied at any point of the table without toppling it.

Question (1955 STEP III Q302)

Four uniform bars AB, BC, CD, DA of length a and weights $w, 2w, w, 2w$ respectively are freely jointed at A, B, C and D. A and C are connected by a light inextensible string of length $l < 2a$ and the whole framework is suspended from A. Find the tension in the string.

Question (1957 STEP III Q301)

A uniform solid consists of a cone and a hemisphere fastened together so that their plane faces coincide, the diameter of the hemisphere being equal to that of the base of the cone. Show that if the semi-vertical angle of the cone is greater than 30° the solid will always move to an upright position if placed with the surface of the hemisphere on a horizontal plane.

Question (1951 STEP III Q402)

A thin uniform rigid rod of weight W resting on a rough peg at A and supported from above by a similar peg at B is in equilibrium under its own weight and the reactions of the pegs. If the coefficient of friction is the same at both pegs, show that it cannot be less than $\mu_0 = \frac{AB\tan\theta}{AC+BC}$, where C is the centre of the rod and θ its inclination to the horizontal. An additional weight W' is attached to a certain point D of the rod such that equilibrium is still maintained but with increase of W' is eventually destroyed. Show that the rod will tend to turn or slip as the coefficient of friction is greater than or less than $\tan\theta$.

Question (1944 STEP I Q303)

Prove that, if a finite set of points in space possesses an axis or a plane of symmetry, then the centroid of the points (i.e. the centre of mass for equal masses placed at the points) lies on the axis or in the plane. Deduce that if the points are such that, for each pair PQ of them, there exists an axis or plane of symmetry of the set with respect to which P and Q are images, then all the points lie on the surface of a sphere.

Question (1948 STEP I Q302)

From a thin uniform rod three lengths are cut and pinned together at their ends to form a triangular frame ABC; J is the centre of gravity of the frame, I the incentre of ABC and G its centroid. Prove that J is the incentre of the triangle whose vertices are the mid-points of BC, CA, AB; hence prove that G is a point of trisection of IJ.

Question (1946 STEP II Q407)

A solid hemisphere of radius a is such that the density at distance r from its centre is proportional to $(a-r)^n$. Show that its centre of mass is at distance $3a/2(n+4)$ from the plane face.

Question (1948 STEP II Q207)

A uniform rod is placed with one end on a rough horizontal plane and the other end against a rough vertical face of a box standing on the same horizontal plane. The vertical plane containing the rod is perpendicular to the vertical face of the box against which the rod rests. A string is attached to the mid-point of the rod and is pulled vertically downwards by a gradually increasing force. Find under what conditions the box will topple over.

Question (1946 STEP II Q306)

A wedge is cut from a uniform solid circular cylinder by a plane which makes an angle α with the base of the cylinder and which touches at a point O the circular boundary of the base. Prove that the mass-centre of the wedge is at a distance $\frac{3}{8}(4+\tan^2\alpha)a$ from O where a is the radius of the cylinder, and explain why this distance does not tend to a as $\alpha \to 0$.

Question (1947 STEP II Q306)

The framework $ABCDEFGH$ consists of eight equal uniform heavy rods smoothly jointed at their ends, and is maintained in the shape of a regular octagon by light struts AF, AG, BD and BE. The framework is in equilibrium, suspended from the mid-point of AB. State which (if any) of the stresses in the struts are thrusts and which tensions, and determine the ratio of the magnitudes of the stresses in AF and AG.

Question (1944 STEP III Q102)

A solid sector is cut out from a uniform solid sphere, of radius a, by a cone of semi-angle β whose vertex is at the centre. Show that the mass-centre of this sector is distant $\frac{3}{8}a(1+\cos\beta)$ from the vertex. If this solid rests in equilibrium with its spherically curved surface in contact with a rough plane which is inclined at an angle β to the horizontal, show that the axis of symmetry is inclined at an angle $\sin^{-1}(\frac{2}{3}\tan\frac{1}{2}\beta)$ to the vertical.

Question (1944 STEP III Q208)

Find an expression for the kinetic energy of n particles of masses m_i $(i = 1, 2, \ldots, n)$ moving in a plane in terms of the velocity V of their centre of mass G and of the velocities of the particles relative to G. Prove also that the kinetic energy is equal to

$$\tfrac{1}{2}\{\Sigma m_i V^2 + \Sigma m_i m_j v_{ij}^2/\Sigma m_i\},$$

where v_{ij} is the velocity of m_i relative to m_j.

Question (1945 STEP III Q202)

AB is a diameter of a solid uniform sphere of radius a and O is the centre. Find the distances from the centre of the sphere of the centres of gravity of the two parts into which the sphere is divided by a plane bisecting BO perpendicularly. Find also the position of the centre of gravity of the part of the sphere contained within the cone obtained by joining A to the circle in which the plane cuts the sphere.

Question (1948 STEP III Q202)

A fixed open cylindrical jar whose radius is a stands on a horizontal table. A smooth uniform rod of length $2l$ $(l > a)$ rests over the rim of the jar with one end pressing against the vertical interior surface of the jar. Prove that in the position of equilibrium the inclination θ of the rod to the horizontal is given by the equation

$$l\cos^3\theta = 2a.$$

Prove that the rod will tumble out of the jar if the inclination of the rod be less than this value of θ.

Question (1948 STEP III Q203)

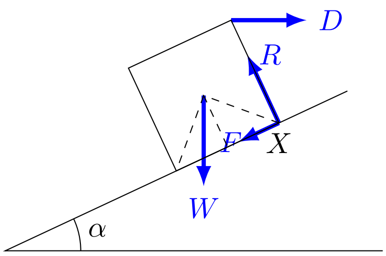
A uniform triangular table with a leg at each corner A, B, C is placed on a rough horizontal plane. Show that the pressure at each point of support is equal to one-third the weight of the table. A gradually increasing couple in a horizontal plane is applied to the table until it begins to turn. Show that the point I about which it begins to turn is such that

$$AI + BI + CI$$

is a minimum. Under what circumstances will the table begin to turn about one of the points of support?

Question (1948 STEP III Q302)

A uniform solid cube is at rest on a rough plane (coefficient of friction μ) inclined at an angle α to the horizontal, two of the edges of the cube being along lines of greatest slope of the plane. A slowly increasing horizontal force is then applied at the middle point of, and perpendicular to, the highest horizontal edge of the cube so that the cube tends to move up the plane. Find how equilibrium will be broken (i) if $\mu = 0.4$, and (ii) if $\mu = 0.25$; explain why the result in case (ii) does not depend on α.



On the point of sliding, we have $F = \mu R$,

$$\begin{aligned}
\text{N2}(\nearrow): \quad & D\cos\alpha - F - W\sin\alpha = 0 \\
& D\cos\alpha - W\sin\alpha = \mu R \\
\text{N2}(\nwarrow): \quad & R - W\cos\alpha - D\sin\alpha = 0 \\
& R = W\cos\alpha + D\sin\alpha \\
\Rightarrow \quad & D\cos\alpha - W\sin\alpha = \mu(W\cos\alpha + D\sin\alpha)
\end{aligned}$$

$$\begin{aligned} D - W\tan\alpha &= \mu(W + D\tan\alpha) \\ D(1-\mu\tan\alpha) &= W(\mu+\tan\alpha) \\ \Rightarrow \qquad D &= \left(\frac{\mu+\tan\alpha}{1-\mu\tan\alpha}\right)W \end{aligned}$$

On the point of toppling R will be acting at X

$$\begin{aligned} \overset{\frown}{X}: \qquad 0 &= D\cos\alpha l - W\cos(45^\circ-\alpha)\frac{l}{\sqrt{2}} \\ \Rightarrow \qquad \cos\alpha D &= \frac{1}{2}\left(\cos\alpha+\sin\alpha\right)W \\ \Rightarrow \qquad D &= \frac{1}{2}\left(1+\tan\alpha\right)W \end{aligned}$$

Therefore we topple before sliding if $D_{\text{top}} < D_{\text{slide}}$, ie

$$\begin{aligned} \frac{1}{2}\left(1+\tan\alpha\right)W &< \left(\frac{\mu+\tan\alpha}{1-\mu\tan\alpha}\right)W \\ \Rightarrow \qquad \frac{1}{2}(1+\tan\alpha) &< \frac{\mu+\tan\alpha}{1-\mu\tan\alpha} \\ \mu=\tfrac{2}{5}: \qquad \frac{1}{2}(1+\tan\alpha) &< \frac{2+5\tan\alpha}{5-2\tan\alpha} \\ \Rightarrow \qquad (5-2\tan\alpha)(1+\tan\alpha) &< 4+10\tan\alpha \\ \Rightarrow \qquad 5+3\tan\alpha-2\tan^2\alpha &< 4+10\tan\alpha \\ \Rightarrow \qquad 1-7\tan\alpha-2\tan^2\alpha &< 0 \\ \Rightarrow \qquad \tan\alpha &< \frac{\sqrt{49+8}-7}{4} = \frac{\sqrt{57}-7}{4} \end{aligned}$$

$$\begin{aligned} \mu=\tfrac{1}{4}: \qquad \frac{1}{2}(1+\tan\alpha) &< \frac{1+4\tan\alpha}{4-\tan\alpha} \\ 4+3\tan\alpha-\tan^2\alpha &< 2+8\tan\alpha \\ 0 &< -2+5\tan\alpha+\tan^2\alpha \\ \tan\alpha &> \frac{-5\pm\sqrt{25+8}}{2} = \frac{\sqrt{33}-5}{2} > \frac{1}{4} \end{aligned}$$

For the cube to be at rest before the cube starts sliding, we must have $\tan\alpha < \mu$, therefore for all values of α the cube always slides

Question (1948 STEP III Q303)

A framework $ABCD$ of four uniform rods each of length a and weight w smoothly jointed together hangs in equilibrium under gravity with AB held in a horizontal position, B and D being joined by a light string of length b ($< \sqrt{2}a$). By the principle of Virtual Work, or otherwise, find the tension in the string.

Question (1944 STEP III Q401)

$n-1$ particles are attached to a light inextensible string A_0A_n at points $A_1, A_2, \ldots, A_{n-1}$, where $A_0A_1 = A_1A_2 = \cdots = A_{n-1}A_n$. A small ring of weight w_n attached to the string at A_n can move without friction on a fixed vertical wire OA_n. The system is in equilibrium with the end A_0 fixed at the same horizontal level as O and

$$OA_0 = OA_1 = OA_2 = \cdots = OA_n.$$

Obtain in terms of r and n the ratio of the weight of the particle at A_r to the total weight. By considering the position of the centre of gravity, show that

$$\sum_{r=1}^{n} \cos\frac{r\pi}{2n} \operatorname{cosec}(2r+1)\frac{\pi}{4n} \operatorname{cosec}(2r-1)\frac{\pi}{4n} = \operatorname{cosec}\frac{\pi}{4n}\left(\operatorname{cosec}\frac{\pi}{4n} - \sec\frac{\pi}{4n}\right).$$

Question (1947 STEP III Q403)

A system of three uniform rods AB, BC, CD of unequal lengths freely jointed at B and C is suspended freely from points A and D not necessarily on the same horizontal level. If the weights of AB and CD are equal, show that in equilibrium the mid-point of BC is at its lowest possible position.

Question (1920 STEP I Q103)

A uniform disc 16 inches in diameter and 1 inch thick weighs 56 lbs. Small masses of 8, 7, 6 ... 1 oz. are fixed on the upper face at distances of 1, 2, 3 ... 8 inches respectively from the centre at equal angular intervals of 45° round the disc, viz. in the N., N.E., E. ... N.W. directions respectively. If the disc is then suspended from the centre of its upper face, what point of the disc will be lowest and by how much will it be depressed? Graphical methods may be used.

Question (1922 STEP I Q105)

Out of a hollow shell bounded by concentric spherical surfaces a hollow ring is cut by two parallel planes. Show that the centres of gravity of the volume of the ring and of its whole surface, plane and curved, are coincident.

Question (1936 STEP I Q104)

Find the centre of mass of a thin uniform wire of length l bent into an arc of a circle of radius a. ABC is a uniform semi-circular wire, of weight w per unit length, and rests in a vertical plane with AC horizontal and B in contact with the ground. Find expressions for the bending moment and the shearing force at any point of the wire.

Question (1939 STEP I Q103)

A uniform lamina in the form of a sector of a circle, of radius a, is bounded by radii that enclose an angle 2β. Prove that the mass centre is distant $(2a \sin\beta)/3\beta$ from the centre of the circle.

A wedge-shaped portion of angle 2β is cut from a uniform solid sphere of radius r by two planes which pass through a diameter. Show that the distance of the mass centre of the wedge from this diameter is

$$\frac{3\pi a \sin\beta}{16\beta}.$$

(Note: the paper has a in the formula, but r is the radius of the sphere. Assuming r is correct.)

$$\frac{3\pi r \sin\beta}{16\beta}.$$

Question (1925 STEP I Q106)

Shew that a system of particles has one and only one centre of mass. Find the centres of mass of the following solids of uniform density, (1) a tetrahedron; (2) a pyramid with a plane base. A solid is bounded by five faces, namely, a parallelogram $ABCD$, a trapezium $ABFE$ in which AB, EF are parallel, a trapezium $CDEF$, a triangle AED and a triangle BFC. Prove that the centre of mass of the solid divides the join of the middle point of EF and the centre of the parallelogram $ABCD$ in the ratio $3AB + EF : AB + EF$.

Question (1927 STEP I Q110)

A particle moves in a plane under the action of a given system of forces; establish the 'principle of work,' namely, that the increase in the kinetic energy of the particle, during any time, is equal to the work done by the forces during that time. Explain clearly under what circumstances the particle may be said to possess a 'potential energy,' and shew that, in this case, the 'principle of work' becomes the 'principle of conservation of energy.' Examine the conception of 'angular momentum' and find an expression for the rate of change of angular momentum of the particle, about a fixed point in the plane of motion. A system consists of two particles of masses m_1 and m_2 and velocities v_1 and v_2. Shew that the kinetic energy of the system is equal to the kinetic energy of the motion relative to the centre of gravity of the system together with the kinetic energy of a mass $m_1 + m_2$ moving with the velocity of the centre of gravity. Express the angular momentum of the system about any fixed point in terms of the angular momentum relative to the centre of gravity and the velocity of the centre of gravity.

Question (1928 STEP I Q106)

Define the centre of mass of a system of particles. Prove that, if G be the centre of mass of a set of n particles of masses $m_1, m_2, \dots m_n$ at the points $A_1, A_2, \dots A_n$, and O be any point:

(i) $\sum_{r=1}^{n} m_r \vec{OA_r} = \vec{OG} \sum_{r=1}^{n} m_r$, where the lengths are added vectorially; and

(ii) $\sum_{r=1}^{n} m_r OA_r^2 = \sum_{r=1}^{n} m_r GA_r^2 + OG^2 \sum_{r=1}^{n} m_r$.

A light equilateral triangular frame is loaded at the corners with weights w_1, w_2, w_3 and suspended from a fixed point by strings of lengths l_1, l_2, l_3 attached to its corners. Prove that the tensions T_1, T_2, T_3 in the strings are given by

$$\frac{w_1 l_1}{T_1} = \frac{w_2 l_2}{T_2} = \frac{w_3 l_3}{T_3} = \frac{\{(w_1 + w_2 + w_3)(w_1 l_1^2 + w_2 l_2^2 + w_3 l_3^2) - (w_2 w_3 + w_3 w_1 + w_1 w_2)a^2\}^{\frac{1}{2}}}{w_1 + w_2 + w_3},$$

where a is the length of a side of the triangle.

Question (1928 STEP I Q109)

Prove that in the motion of a system of particles in one plane:

(i) the increase of kinetic energy in any time is equal to the work done by the forces in that time;

(ii) the centre of mass moves as a particle of mass equal to that of the system under the action of all the forces moved parallel to themselves to act at the centre of mass;

(iii) the motion relative to the centre of mass is independent of the motion of that point.

A uniform rod of length l has one end fastened to a fixed point which is at a height l above the ground. The rod is given an initial angular velocity ω in a vertical plane, starting from rest in its lowest position; and the fastening gives way when the rod first reaches a horizontal position. Shew that, if the rod is horizontal when it strikes the ground, then

$$l\omega^2 = 3g + gk^2\pi^2/(2 + k\pi),$$

where k is a positive integer.

Question (1919 STEP I Q110)

Prove the following properties of the centre of mass:

(1) The centre of mass of a system of particles moves as if it were a particle of mass equal to the total mass acted on by all the external forces.

(2) The kinetic energy of the system is equal to the kinetic energy of a particle of mass equal to the total mass moving with the centre of mass plus the kinetic energy of the system relative to the centre of mass.

(3) The increase in kinetic energy of the system relative to the centre of mass is equal to the work done by the forces calculated as though the centre of mass were at rest.

Apply these results to find the effect of the recoil of the gun on the range of a projectile, assuming that the same amount of kinetic energy is generated by the explosion in all cases.

Question (1923 STEP I Q205)

Show that the centre of gravity of a uniform semicircular rod is at a distance from the centre equal to $2/\pi$ times the radius. A circular disc, whose mass per unit area is σr, where σ is a constant and r the distance from the centre, is divided into two by a diameter. Find the centre of gravity of either half.

Question (1933 STEP I Q203)

Find the centre of gravity of a uniform solid hemisphere. A solid consists of a hemisphere of radius a from which a sphere of radius $\frac{1}{2}a$ has been removed. It rests with its base on a rough plane inclined at an angle α to the horizontal, and a gradually increasing force is applied at the pole of the hemisphere in a direction up the plane parallel to a line of greatest slope. Shew that the solid slips or tilts according as $3\tan\lambda + 2\tan\alpha$ is less than or greater than 3, where λ is the angle of friction. Find the value of the force required to destroy the equilibrium.

Question (1935 STEP I Q205)

(a) Find the centre of gravity of the portion of a uniform spherical shell contained between two parallel planes. (b) Find the centre of gravity of a triangle formed by three uniform rods of the same material.

Question (1940 STEP I Q202)

Find the position of the centre of mass of a uniform solid bounded by a parabolic cylinder of latus rectum $4a$, by two planes perpendicular to the generators, and a plane perpendicular to the axis of symmetry at a distance h from the vertex.

Shew that the solid will rest in equilibrium on a horizontal plane with its plane of symmetry inclined at an angle $\tan^{-1}\sqrt{\frac{5a}{3h-10a}}$ to the horizontal provided that $3h > 10a$.

Question (1941 STEP I Q202)

Show that the mass centre of a wedge-shaped portion cut from a uniform solid sphere of radius a by two planes inclined at an angle 2α and intersecting in a diameter is distant $3\pi a \sin\alpha/16\alpha$ from that diameter. Find the corresponding result for a wedge cut from a sphere whose density is ρ from $r = 0$ to $r = a$ and σ from $r = a$ to $r = b$. Deduce that the mass centre of a lune, of angle 2α, cut from a thin spherical shell of radius a, is distant $\pi a \sin\alpha/4\alpha$ from the centre of the sphere.

Question (1924 STEP II Q207)

Three spheres, each of radius 3 inches, rest in mutual contact on a horizontal table, and a fourth sphere, of radius 2 inches, rests upon them. Find (i) the height above the table of the highest point of the smaller sphere, and (ii) the inclination to the horizontal of a plane which touches the smaller sphere and two of the larger ones.

Question (1920 STEP II Q307)

Prove that, (i) the centre of inertia of a uniform triangular lamina is the same as that of three equal particles placed at the angular points, (ii) the centre of inertia of a uniform quadrilateral lamina is the same as that of four equal particles placed at the angular points and of an equal negative particle placed at the intersection of the diagonals. Find the ratio of the masses of three particles placed at the angular points of a triangle such that their centre of inertia is at the orthocentre of the triangle.

Question (1918 STEP III Q305)

Find the centroids of an arc and a sector of a circle. Shew that the centroid of a segment of a circle divides the line of symmetry into parts in the ratio of 2:3, if the breadth of the segment is small compared with the radius.

Question (1918 STEP III Q307)

Find the velocity of the centre of inertia of two particles whose masses and velocities are given. Shew that the kinetic energy of two masses is equal to that of the sum of the masses moving with the velocity of the centre of inertia together with that of each mass moving with its velocity relative to the centre of inertia.

Question (1919 STEP IIII Q302)

From the points B, C, D of a light string ABCDE weights proportional to 4, 8 and 5 are hung respectively. It is found that the portions of the string BC and CD make angles of 25° and 15° respectively with the horizontal. Find graphically the angles which the strings AB and DE make with the vertical.

None

Question (1941 STEP I Q401)

Define the *Centre of Mass* of a system of particles and shew that the point is unique. A particle at O is subject to forces given in magnitude and direction by

$$k_1.\vec{OP}_1, k_2.\vec{OP}_2, \dots k_n.\vec{OP}_n,$$

where $P_1, P_2, \dots P_n$ are definite points. Shew how the resultant can be calculated by finding a centre of mass. A rigidly connected system of particles of masses m_1 at O_1, m_2 at $O_2, \dots$ and m_l at O_l is subject to a system of forces such that the particle m_s at O_s is acted upon by a force $m_s k_r.\vec{O_s P_r}$ towards P_r, where r takes values 1 to n, and s takes values 1 to l. Shew that the resultant is given by $MK.\vec{OP}$, where O is the centre of mass of m_1 at $O_1, \dots m_l$ at O_l, P is the centre of mass of k_1 at $P_1, \dots k_n$ at P_n, and where $M = \sum m_s, K = \sum k_r$.

Question (1927 STEP III Q401)

A body of uniform material consists of a solid right circular cone and a solid hemisphere on opposite sides of the same circular base of radius r. Find the greatest possible height of the cone if the body can rest on a horizontal plane in stable equilibrium with the cone uppermost.

Question (1940 STEP III Q410)

O is the centre of a rectangle ABCD. E is the mid-point of CD and F is the mid-point of AD. AB, BC are of length $2a, 2b$ respectively.

A uniform lamina in the shape ABCEFA is of mass M. Find the moment of inertia of the lamina about the line through its mass centre and parallel to AB.

Question (1915 STEP IIII Q402)

Two particles of a system of masses m_1, m_2 are at A, B. If these two particles are interchanged, prove that the centre of gravity of the whole system moves through a distance $\frac{m_1 - m_2}{\Sigma m} AB$ parallel to AB.

Question (1916 STEP II Q509)

Prove that if the sum of the resolutes in a given direction of the external forces on any number of particles be zero, the sum of the momenta of the particles in that direction is constant. Two equal particles A and B are connected by a light inextensible string of length a which is stretched at full length perpendicular to the edge of the table. The particle A is drawn just over the edge of the table and is then released from rest in this position. Describe the nature of the subsequent motion and shew that after B leaves the table the centre of inertia of the two particles describes a parabola of latus rectum $a/2$.

Question (1926 STEP III Q502)

Find the centre of mass of a uniform solid hemisphere. If the hemisphere is suspended by a string passing through a smooth fixed ring and attached to two points on the rim of the plane face at opposite ends of a diameter, show that in equilibrium with the plane face inclined to the horizontal, $8\tan\theta = 3$, where 2θ is the angle between the parts of the string.

Question (1931 STEP III Q510)

Prove that the volume enclosed by rotating a closed plane curve about a non-intersecting coplanar axis is given by the product of the area enclosed by the curve, and the length of the path traced by the centroid of the area. Shew that the volume of the surface formed by rotating the larger part of an ellipse about the latus rectum which is its boundary, is given by

$$2\pi\frac{l^3}{(1-e^2)^3}\left\{e\cos^{-1}(-e)+\sqrt{1-e^2}\left(\frac{2+e^2}{3}\right)\right\},$$

where e is the eccentricity and l the semi-latus rectum of the ellipse.

Question (1932 STEP III Q510)

Prove that the centre of mass of a uniform lamina bounded by part of the parabola $y^2 = 2lx$ and a focal chord of the parabola always lies on the parabola

$$y^2 = \frac{5l}{4}\left(x - \frac{3l}{10}\right)$$

whatever the inclination of the focal chord to the axis of the parabola.

Question (1930 STEP II Q611)

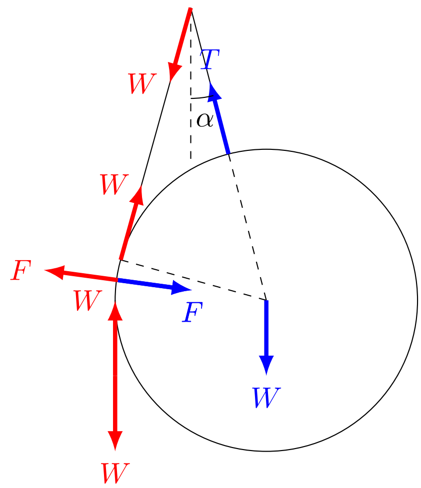
Explain how to determine the position of the centre of mass of a uniform plane lamina which is bounded by the line $\theta = 0$ and an arc of a curve whose polar equation is given. Find the polar coordinates of the centre of mass of one of the halves of the loop of the curve $r = a\cos n\theta$ which is bisected by the line $\theta = 0$.

Question (1913 STEP III Q604)

Investigate the position of the centre of gravity of a homogeneous solid hemisphere. Find the centre of gravity of a semi-circular plate of radius a and of uniform small thickness such that the density at any point varies as $\sqrt{a^2 - r^2}$ where r is the distance of the point from the centre.

Question (1922 STEP III Q601)

A smooth sphere is suspended from a fixed point by a string of length equal to its radius. To the same point a second string is attached which after passing over the sphere supports a weight equal to that of the sphere. Show that the first string then makes an angle $\sin^{-1}(\frac{1}{4})$ with the vertical.



First notice by considering moments about the centre of the sphere, the three forces acting on it are weight (acting at the centre, no moment), force acting from the string (acting towards the centre, no moment) and the string pulling it up. Therefore the line of action of the tension runs through the centre.

Therefore the point where the second string meets the sphere is part of a $30, 60, 90$ triangle.

By considering the red forces on the string, we must also have that the force F is

$$-W\begin{pmatrix} \sin(30^\circ - \alpha) \\ \cos(30^\circ - \alpha) \end{pmatrix} - W\begin{pmatrix} 0 \\ -1 \end{pmatrix} = W\begin{pmatrix} -\sin(30^\circ - \alpha) \\ 1 - \cos(30^\circ - \alpha) \end{pmatrix}$$

Finally, by considering that the system is in equilibrium, we must also have the blue forces sum to zero, ie

Question (1924 STEP III Q603)

Find the position of the centre of gravity of a uniform semicircular disc. If any point P is taken upon the diameter AB of such a disc and the semicircles upon AP, BP as diameters are removed, find the position of centre of gravity of the part remaining. Show that for different positions of P the centre of gravity lies midway between P and a certain fixed point.

Question (1925 STEP III Q614)

Two masses m, m', connected by a weightless rod, lie on a smooth horizontal table. The rod is struck at right angles to its length by an impulsive force F; find the velocities of the masses, and show that the kinetic energy is least if F is applied at the centre of gravity of the masses.

Question (1922 STEP I Q709)

Prove that the centre of gravity of three uniform rods in the form of a triangle coincides with the centre of the circle inscribed in the triangle formed by joining their middle points. Find the position of the centre of gravity of the portion of a solid circular cylinder between its circular base and an oblique section made by a plane touching its base.