

Question (1966 STEP I Q12)

Two planes

$$x - 3y + 2z = 2, \tag{1}$$

$$2x - y - z = 9, \tag{2}$$

meet in the line l . Find the equations of (i) the plane through the origin which contains l , (ii) the plane through the origin which is perpendicular to l . Find also the coordinates of the reflection of the origin in l .

None

Question (1968 STEP I Q14)

The number $a_{11} + a_{22} + a_{33}$ is called the trace of the matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

If \mathbf{A} and \mathbf{B} are two 3×3 matrices, show that the traces of the matrices \mathbf{AB} and \mathbf{BA} are equal. If the matrix \mathbf{AB} represents a rotation through an angle ϕ about the directed axis U and \mathbf{A} represents a rotation interchanging the axes U and V , explain why \mathbf{BA} represents a rotation through the angle ϕ about V . Given that the matrix

$$\mathbf{M} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represents a rotation through the angle ϕ about the z -axis, and that the matrix \mathbf{C} represents a rotation about some axis, find a formula for the angle of rotation in terms of the trace of \mathbf{C} .

None

Question (1974 STEP I Q7)

Let $E^{(ij)}$ be the 3×3 real matrix with 1 in the (i, j) th position and zeros everywhere else. Let $F^{(ij)}(\lambda) = I + \lambda E^{(ij)}$ where I is the identity 3×3 matrix. Show that for an arbitrary 3×3 matrix A , $F^{(ij)}(\lambda)A$ (for $i \neq j$) is the matrix obtained from A by replacing the i th row $A^{(i)}$ by $A^{(i)} + \lambda A^{(j)}$ where $A^{(j)}$ is the j th row of A .

Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 3 & 1 & 4 \\ 0 & 3 & 1 \end{pmatrix}$. Find a matrix Q , which is the product of several $F^{(ij)}(\lambda)$

for suitable i, j and λ , such that QA is of the form $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$. Hence solve the equation

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

None

Question (1969 STEP II Q4)

Show that the triangles in the complex plane with vertices z_1, z_2, z_3 and z'_1, z'_2, z'_3 respectively are similar if

$$\begin{vmatrix} z_1 & z'_1 & 1 \\ z_2 & z'_2 & 1 \\ z_3 & z'_3 & 1 \end{vmatrix} = 0.$$

Discuss whether the converse of this result is true.

None

Question (1971 STEP III Q6)

Prove that, for any four points A, B, C, D in a plane,

$$\begin{vmatrix} 0 & 2AB^2 & AB^2 + AC^2 - BC^2 & AB^2 + AD^2 - BD^2 \\ 2AC^2 & 0 & 2BC^2 & AC^2 + AD^2 - CD^2 \\ AC^2 + AB^2 - CB^2 & 2AB^2 & 0 & 2CD^2 \\ AD^2 + AB^2 - DB^2 & AD^2 + AC^2 - DC^2 & 2CD^2 & 0 \end{vmatrix} = 0$$

None

Question (1964 STEP I Q302)

Three 'ordered triplets'

$$\mathbf{a} = (1, 1, 1), \quad \mathbf{b} = (1, 2, 3), \quad \mathbf{c} = (1, 3, 6)$$

are given, each consisting of three numbers in an assigned order. [Thus the triplets $(3, 5, 7)$, $(3, 7, 5)$ are different.] By a 'combination'

$$\lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c}$$

is meant the triplet

$$(\lambda + \mu + \nu, \lambda + 2\mu + 3\nu, \lambda + 3\mu + 6\nu).$$

Prove that values λ, μ, ν can be found so that the combination is the given triplet

$$\mathbf{x} = (p, q, r)$$

and find λ, μ, ν in terms of p, q, r . Express \mathbf{a} as a combination of $\mathbf{b}, \mathbf{c}, \mathbf{x}$ in the form $\alpha \mathbf{b} + \beta \mathbf{c} + \gamma \mathbf{x}$, or, in detail,

$$(\alpha + \beta + \gamma p, 2\alpha + 3\beta + \gamma q, 3\alpha + 6\beta + \gamma r),$$

stating any condition that may be necessary for this form of expression to be possible.

None

Question (1960 STEP III Q304)

A , B and C are the three angles of a triangle. Show that

$$\begin{vmatrix} \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \\ \sin^3 A & \sin^3 B & \sin^3 C \end{vmatrix} = 0.$$

None

Question (1960 STEP II Q102)

The coordinates of any point on a curve are given by $x = \phi(t)$, $y = \psi(t)$, where t is a parameter; prove that the equation of the tangent is

$$\begin{vmatrix} x & \phi(t) & \phi'(t) \\ y & \psi(t) & \psi'(t) \\ 1 & f(t) & f'(t) \end{vmatrix} = 0.$$

Prove that the condition that the tangents at the points of the curve

$$x = at/(t^2 + bt^2 + ct + d), \quad y = a/(t^2 + bt^2 + ct + d),$$

whose parameters are t_1, t_2, t_3 may be concurrent is

$$3(t_2t_3 + t_3t_1 + t_1t_2) + 2b(t_1 + t_2 + t_3) + b^2 = 0.$$

None

Question (1959 STEP II Q304)

By the 'first octant' of 3-dimensional space with a given co-ordinate system we mean the set of points (x, y, z) with

$$x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

A line λ passes through the origin and contains no other point of the first octant. Show that there is a plane π which passes through λ and contains no point of the first octant except the origin.

None

Question (1953 STEP III Q106)

Show that the condition that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

should represent two straight lines is

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

Show further that necessary conditions that these lines should be real are $h^2 \geq ab$; $f^2 \geq bc$; $g^2 \geq ca$. If $\Delta = 0$, prove that the point of intersection of the lines is

$$(hf - bg)/(ab - h^2); \quad (gh - af)/(ab - h^2).$$

Question (1936 STEP I Q110)

Two triangles $ABC, A'B'C'$ in a plane are such that AA', BB', CC' are concurrent in a point O . $BC, B'C'$ meet in L ; $CA, C'A'$ in M , and $AB, A'B'$ in N . Prove that L, M, N are collinear. Show further that there exists a unique conic S with respect to which the triangles reciprocate into each other, and that the polar of O with respect to S is the line LMN .

Question (1914 STEP III Q211)

Through any point P lines are drawn parallel to the internal bisectors of the angles of a triangle ABC to meet the opposite sides in D, E, F . Prove that if D, E, F are collinear P lies on the conic

$$(b + c)\beta\gamma + (a + c)\alpha\gamma + (a + b)\alpha\beta = 0,$$

where the coordinates are trilinear and ABC is the triangle of reference. Prove that the centre of the conic is the centre of the inscribed circle of the triangle whose vertices are the mid-points of the sides of ABC .

Question (1924 STEP I Q506)

Prove that the two straight lines $x^2(\tan^2 \theta + \cos^2 \theta) - 2xy \tan \theta + y^2 \sin^2 \theta = 0$ form with the line $x = c$ a triangle of area c^2 .

Question (1931 STEP II Q504)

Obtain the condition that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

shall represent some pair of straight lines l_1, l_2 . If also the equation $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0$ represents a pair of straight lines l_2, l_3 (so that l_2 is a line common to the two pairs), shew that

$$(a_1c - ac_1)^2 = 4(g_1c - gc_1)(a_1g - ag_1),$$

and $(b_1c - bc_1)^2 = 4(f_1c - fc_1)(b_1f - bf_1).$

Shew further that the coordinates of the point of intersection of l_1 and l_3 are

$$\left(\frac{(bc_1 - cb_1)(ca_1 - ac_1)}{2(ab_1 - ba_1)(cg_1 - gc_1)}, \frac{(bc_1 - cb_1)(ca_1 - ac_1)}{2(ab_1 - ba_1)(fc_1 - cf_1)} \right).$$

Question (1914 STEP II Q601)

If

$$\begin{aligned} x^2 - yz + (a - \lambda)x &= 0, \\ y^2 - zx + (b - \lambda)y &= 0, \\ z^2 - xy + (c - \lambda)z &= 0, \end{aligned}$$

and

$$x^2y^2 + y^2z^2 + z^2x^2 = xyz(x + y + z),$$

prove that

$$3\lambda = a + b + c,$$

and that

$$a^2 + b^2 + c^2 = bc + ca + ab.$$