Question (1966 STEP I Q3)

Evaluate

(i) $\sum_{r=1}^{n} \frac{r-1}{r(r+1)(r+2)} \quad (n \geq 3)$;

(ii) $1^2 + 2^2 c_1 + 3^2 c_2 + \ldots + (n+1)^2 c_n$, where $c_r = \frac{n!}{r!(n-r)!}$.

Question (1968 STEP I Q3)

The equality

$$\frac{ax^2 + bx + c}{(x+\alpha)(x+\beta)(x+\gamma)} = \frac{A}{(x+\alpha)} + \frac{B}{(x+\beta)} + \frac{C}{(x+\gamma)},$$

in which $a, b, c, \alpha, \beta, \gamma$ are real numbers, holds for all real x (other than $-\alpha, -\beta, -\gamma$). Find a necessary and sufficient condition, in terms of a, b, c, for $A + B + C$ to vanish. Evaluate

$$\sum_{n=1}^{N} \frac{2n+5}{n(n+1)(n+3)} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{2n+5}{n(n+1)(n+3)}.$$

Question (1977 STEP I Q1)

Given that, for all x,

$$\frac{ax^2 + bx + c}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma},$$

find the condition that $A + B + C = 0$. Hence, or otherwise, evaluate

$$\sum_{n=1}^{N} \frac{3n+1}{n(n+1)(n+2)}.$$

Proof 1: Let $f(x) = \frac{ax^2+bx+c}{(x-\alpha)(x-\beta)(x-\gamma)}$ then $\lim_{x\to\infty} xf(x) = a$ but also $A+B+C$, therefore $A + B + C = a$. Therefore $A + B + C = 0 \Leftrightarrow a = 0$

Proof 2: Multiply both sides by $(x-\alpha)(x-\beta)(x-\gamma)$ then the coefficient of x^2 is a on the LHS and $A + B + C$ on the RHS, therefore $A + B + C = a$

$$\begin{aligned}
& & \frac{3n+1}{n(n+1)(n+2)} &= \frac{1}{2n} + \frac{2}{n+1} - \frac{5}{2(n+2)} \\
\Rightarrow & & \sum_{n=1}^{N} \frac{3n+1}{n(n+1)(n+2)} &= \sum_{n=1}^{N} \left(\frac{1}{2n} + \frac{2}{n+1} - \frac{5}{2(n+2)} \right) \\
& & &= \quad \frac{1}{2\cdot 1} + \frac{2}{2} - \frac{5}{2\cdot 3} + \cdots \\
& & &\quad + \frac{1}{2\cdot 2} + \frac{2}{3} - \frac{5}{2\cdot 4} + \cdots \\
& & &\quad + \frac{1}{2\cdot 3} + \frac{2}{4} - \frac{5}{2\cdot 5} + \cdots \\
& & &\quad \cdots
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2\cdot(N-2)}+\frac{2}{N-1}-\frac{5}{2\cdot N}+\cdots\\
&+\frac{1}{2\cdot(N-1)}+\frac{2}{N}-\frac{5}{2\cdot(N+1)}+\cdots\\
&+\frac{1}{2\cdot N}+\frac{2}{N+1}-\frac{5}{2\cdot(N+2)}+\cdots\\
=&\frac{3}{2}+\frac{2}{N+1}-\frac{5}{2(N+1)}-\frac{5}{2(N+2)}\\
=&\frac{3}{2}-\frac{1}{2(N+1)}-\frac{5}{2(N+2)}
\end{aligned}
$$

Question (1978 STEP I Q6)

Decompose

$$\frac{3x^2+2ax+2bx+ab}{x^3+(a+b)x^2+abx}$$

into partial fractions. By considering the smallest denominator or otherwise, show that this expression takes the value 1 for only a finite number of positive integral values of x, a and b (You are not required to find all such values.)

Question (1978 STEP I Q13)

Express the function

$$f(x)=\frac{x^3-x}{(x^2-4)^2}$$

in partial fractions with constant numerators. Find the nth derivative of $f(x)$ at $x=0$.

None

Question (1979 STEP III Q7) (i) Calculate

$$\sum_{j=1}^{n-1}\frac{n-2j}{j(n-j)}.$$

(ii) Suppose that $a_1 < a_2 < \ldots < a_n$ and that $b_1, b_2, \ldots, b_n$ are distinct real numbers. Let $c_1, \ldots, c_n$ be the numbers $b_1, \ldots, b_n$ arranged in increasing order, and let $d_1, \ldots, d_n$ be the numbers $b_1, \ldots, b_n$ arranged in decreasing order. Show that

$$\sum_{i=1}^{n}a_id_i \le \sum_{i=1}^{n}a_ib_i \le \sum_{i=1}^{n}a_ic_i.$$

Question (1960 STEP I Q104)

Sum the series

$$\sum_{1}^{q} \frac{1}{n(n+1)}.$$

Prove that

$$\frac{1}{p+1} - \frac{1}{q+1} < \sum_{p+1}^{q} \frac{1}{n^3} < \frac{1}{p} - \frac{1}{q}.$$

If S denotes the sum in the middle, and A denotes the average of the two bounds, prove that

$$\frac{1}{3}\left\{\frac{1}{p(p+1)(p+2)} - \frac{1}{q(q+1)(q+2)}\right\} < A - S < \frac{1}{3}\left\{\frac{1}{(p-1)p(p+1)} - \frac{1}{(q-1)q(q+1)}\right\}.$$

Question (1963 STEP I Q103)

Express in partial fractions

$$\frac{(x+1)(x+2)\cdots(x+n+1)-(n+1)!}{x(x+1)(x+2)\cdots(x+n)}.$$

Hence, or otherwise, prove that

$$\frac{c_1}{1} - \frac{c_2}{2} + \frac{c_3}{3} - \ldots + (-1)^{n-1}\frac{c_n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n},$$

where $c_r = n!/r!(n-r)!$.

Notice that deg of the numerator is equal to the degree of the denominator, so:

$$\frac{(x+1)(x+2)\cdots(x+n+1)-(n+1)!}{x(x+1)(x+2)\cdots(x+n)} = 1 + \frac{A_0}{x} + \frac{A_1}{x+1} + \cdots + \frac{A_n}{x+n}$$

$$\begin{aligned}
\Rightarrow \quad & (x+1)\cdots(x+n+1)-(n+1)! = x(x+1)\cdots(x+n) + A_0(x+1)\cdots(x+n) + \cdots + A \\
x = 0: \quad & 0 = A_0 \\
x = -k: \quad & 0-(n+1)! = A_k(-k)\cdots(-1)(1)\cdots(n-k) \\
\Rightarrow \quad & 0-(n+1)! = A_k(-1)^k k!(n-k)! \\
\Rightarrow \quad & A_k = (-1)^{k+1}\frac{n!}{k!(n-k)!}(n+1) \\
\Rightarrow \quad & \frac{(x+1)(x+2)\cdots(x+n+1)-(n+1)!}{x(x+1)(x+2)\cdots(x+n)} = 1 + (n+1)\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\frac{1}{x+k}
\end{aligned}$$

This expression is true if $x \neq 0, -1, -2, \cdots, -n$. Notice however the LHS and RHS are both well-defined when $x = 0$. Therefore we plug in 0 to see:

$$1 + (n+1)\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\frac{1}{k} = \frac{(n+1)!\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1}\right)}{n!}$$

$$= (n+1)\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}\right)$$

$$\Rightarrow \qquad \frac{1}{n+1}+\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\frac{1}{k} = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}$$

$$\Rightarrow \qquad (-1)^{k+1}\binom{n}{k}\frac{1}{k} = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}$$

Question (1959 STEP IIII Q304)

If

$$f(a,b) = \sum_{n=1}^{\infty}\frac{1}{n^2+an+b},$$

show that

$$f(a+\beta, a\beta) = \frac{1}{\beta-a}\left(\frac{1}{a+1}+\frac{1}{a+2}+\ldots+\frac{1}{\beta}\right)$$

whenever $\beta - a$ is a positive integer and a is not a negative integer. Evaluate $f(-\frac{1}{2}, 0)$.

Question (1960 STEP IIII Q303)

A sequence of integers a_n is defined by

$$a_1 = 2,$$

$$a_{n+1} = a_n^2 - a_n + 1 \quad (n > 0).$$

Prove:

(i) $\sum_{n=1}^{\infty} \frac{1}{a_n} = 1$;

(ii) If $m \neq n$, then a_m and a_n have no common factor greater than 1.

Question (1964 STEP II Q201)

(i) In the equation

$$\frac{k_1}{x-a_1}+\frac{k_2}{x-a_2}+\ldots+\frac{k_n}{x-a_n} = 0$$

the numbers k_i are positive and the a_i are distinct real numbers. Prove that the roots of the equation are all real. (ii) Find necessary and sufficient conditions for the equation

$$2x^5 - 5px^2 + 3q = 0,$$

where p and q are positive, to have (a) one, (b) three real roots.

Question (1957 STEP I Q102)

By putting the expression

$$\frac{(x+1)(x+2)\dots(x+n)}{x(x-1)(x-2)\dots(x-n)}$$

into partial fractions, or otherwise, prove that the system of n equations

$$\sum_{r=0}^{n} \frac{X_r}{r+s} = 0 \quad (s = 1, 2, \dots, n)$$

in the $n+1$ unknowns $X_0, X_1, \dots, X_n$ is satisfied by

$$X_r = \frac{(-1)^{n-r}(n+r)!}{(r!)^2(n-r)!} \quad (r = 0, 1, \dots, n).$$

Show also that, with these values,

$$\sum_{r=0}^{n} X_r = 1.$$

Question (1952 STEP IIII Q303)

$f(x)$ is a polynomial of degree n. If $a_1, \dots, a_n$ are distinct and

$$\frac{f(x)}{(x-a_1)^2(x-a_2)\dots(x-a_n)} = \frac{A_0}{(x-a_1)} + \frac{A_1}{(x-a_1)^2} + \frac{A_2}{(x-a_2)} + \dots + \frac{A_n}{(x-a_n)},$$

find $A_0, \dots, A_n$. Find the polynomial of the fourth degree such that $f(0) = f(1) = 1, f(2) = 13, f(3) = 73, f'(0) = 0$.

Question (1955 STEP II Q201)

(i) If $a_1 < a_2 < \dots < a_n$ and $0 < A_1, A_2, \dots, A_n$, prove that the zeros of the rational function $\displaystyle\sum_{i=1}^{n} \frac{A_i}{x-a_i}$ are all real.

(ii) If $a_1 < a_2 < \dots < a_n$ and $0 < A_1, A_2, \dots, A_{k-1}, A_{k+1}, \dots, A_n$, for some k such that $1 < k < n$, and if

$$A_1 + A_2 + \dots + A_n < 0,$$

prove that the zeros of the rational function $\displaystyle\sum_{i=1}^{n} \frac{A_i}{x-a_i}$ are all real.

(iii) If $x_1 = 0$ and $\displaystyle\sum_{\substack{l=1 \\ l \neq k}}^{n} \frac{1}{x_k - x_l} = 1$, for $k = 2, 3, \dots, n$, where all of $x_1, x_2, \dots, x_n$ are different, prove that $x_2, \dots, x_n$ are real and positive.

None

Question (1944 STEP IIII Q303)

The polynomial $f(x)$ has only simple zeros $a_1, a_2, \ldots, a_n$. Show that, if

$$\frac{1}{[f(x)]^2} = \sum_{i=1}^{n} \frac{A_i}{(x-a_i)^2} + \frac{B_i}{x-a_i},$$

then

$$A_i = \frac{1}{[f'(a_i)]^2}, \quad B_i = -\frac{f''(a_i)}{[f'(a_i)]^3}.$$

Hence, or otherwise, express

$$\frac{1}{(x^{2n}-1)^2},$$

where n is a positive integer, as the sum of real partial fractions.

Question (1946 STEP IIII Q301)

Express

$$f(x) = \frac{x+1}{(x+2)(x-1)^2}$$

in partial fractions. Show that the coefficient of x^n in the expansion of $f(x)$ in increasing powers of x is

$$\frac{1}{9}\{12n + 10 - (-)^n\}.$$

Question (1946 STEP II Q403)

(a) Express the function $\frac{x^2-2}{(x^2+x+2)^2(x^2+x+1)}$ as partial fractions in the form

$$\frac{Ax+B}{(x^2+x+2)^2} + \frac{Cx+D}{x^2+x+2} + \frac{Ex+F}{x^2+x+1},$$

determining the values of A, B, C, D, E and F. (b) Show, if n is a positive integer, that in the expansion of $\frac{x-1}{(x-2)^n(x-3)}$ in partial fractions, the numerator of the fraction $\frac{1}{(x-2)^r}$ is -2 if $n > r > 1$. What is it when $r = n$?

Question (1945 STEP II Q204)

Express

$$\frac{ax^2 + 2bx + c}{(x-\alpha)^2(x-\beta)^2}$$

in partial fractions, when all the coefficients are general and not subject to any conditions. Find the condition that

$$\int \frac{ax^2 + 2bx + c}{(Ax^2 + 2Bx + C)^2}\, dx$$

should be a rational function (as defined in Question A 2), when the two quadratic expressions have no common factor and $B^2 > AC$.

Question (1928 STEP I Q103)

Express

$$\frac{57x^3 - 25x^2 + 9x - 1}{(x-1)^2(2x-1)(5x-1)}$$

as a sum of partial fractions; and expand in ascending powers of x as far as the term in x^4.

Question (1936 STEP I Q103)

Express the function

$$f(x) = \frac{x^3 - x}{(x^2-4)^2}$$

in partial fractions (with numerical numerators). Find the value of the nth derivative of $f(x)$ for $x = 0$.

Question (1939 STEP I Q102)

Express

$$y = \frac{4}{(1-x)^2(1-x^2)}$$

in partial fractions. Show that, when $x = 0$, the value of

$$\frac{1}{n!}\frac{d^n y}{dx^n}$$

is equal to $(n+2)^2$ when n is even, and $(n+1)(n+3)$ when n is odd.

Question (1924 STEP I Q106)

It is given that

$$k_1/(x-a_1) + k_2/(x-a_2) + \cdots + k_n/(x-a_n) = 0,$$

where $k_1 + k_2 + \cdots + k_n = 0$. Prove that, if $x = (py+q)/(ry+s)$, $a_1 = (pb_1 + q)/(rb_1+s), \ldots, a_n = (pb_n+q)/(rb_n+s)$, where $ps - qr \neq 0$, then

$$k_1/(y-b_1) + \cdots + k_n/(y-b_n) = 0.$$

Question (1920 STEP I Q102)

Prove that, if P and Q are two given polynomials in x, with no common factor, it is possible to find two other polynomials A and B such that

$$AP + BQ = 1.$$

Prove further that, if (A_1, B_1) is one solution of the problem, the most general solution is $(A_1 + CQ, B_1 - CP)$, where C is any polynomial, and that, if A is restricted to be of lower order than Q, there is one and only one solution. Hence (or otherwise) shew that, if $f(x), \phi(x)$ are two polynomials of order m, n respectively, with no common factor, and $\phi = (x-a)^p(x-b)^q \ldots$, where $a, b, \ldots$ are different, and $p, q, \ldots$ are positive integers, then $f(x)/\phi(x)$ can be expressed in the form

$$R(x) + A_1/(x-a) + \cdots + A_p/(x-a)^p + B_1/(x-b) + \cdots + B_q/(x-b)^q + \ldots,$$

where $A_1, \ldots B_1, \ldots$ are constants, and R is a polynomial of order $m-n$ if $m \geq n$ but otherwise zero.

Question (1942 STEP I Q103)

If $P(x), Q(x)$ are polynomials in x with a highest common factor $H(x)$, shew that polynomials $A(x), B(x)$ can be found such that $AP + BQ = H$ identically. Hence shew that $F(x)/G(x)$, where the degree of F is less than that of G, may be expressed as the sum of real partial fractions. Express in partial fractions

$$\frac{x}{(x+2)(x-1)^n},$$

where n is a positive integer.

Question (1916 STEP II Q202)

If

$$\frac{1}{(x+1^2)(x+2^2)\dots(x+n^2)} = \frac{A_1}{x+1^2} + \frac{A_2}{x+2^2} + \cdots + \frac{A_n}{x+n^2},$$

show that

$$A_r = \frac{(-1)^{r-1}2r^2}{(n-r)!(n+r)!},$$

where 0! is taken to mean unity. Hence show that if p is any integer $< n$,

$$\frac{1^{2p}}{(n-1)!(n+1)!} - \frac{2^{2p}}{(n-2)!(n+2)!} + \cdots + \frac{(-1)^{n-1}n^{2p}}{(2n)!} = 0.$$

Question (1920 STEP II Q201)

Resolve

$$\frac{1}{(1-x)^2(1+x^2)}$$

into partial fractions. Prove that, if this function is expanded in a series of ascending powers of x, the coefficient of x^{4n} is $2n+1$ and each of the coefficients of x^{4n-3}, x^{4n-2}, and x^{4n-1} is $2n$.

None

Question (1923 STEP II Q201)

Resolve into partial fractions

$$\frac{3x^2-6x+2}{(x^2+1)(x-3)^2}.$$

Question (1934 STEP II Q201)

State and prove a rule for expressing

$$\frac{P(x)}{Q(x)}$$

as the sum of a polynomial and partial fractions, where P and Q are polynomials, and Q has no repeated factors.

Express in this form

$$\frac{(x-a)(x-b)(x-c)(x-d)}{(x+a)(x+b)(x+c)(x+d)},$$

(i) when a, b, c, d are all unequal, (ii) when they are all equal.

Question (1913 STEP IIII Q206)

Discuss the expression of a rational function of x as the sum of a polynomial and of partial fractions whose denominators are powers of linear or quadratic functions of x and numerators respectively constants or linear functions of x. Illustrate by the cases

$$\text{(i)}\ \frac{x^5}{(x+1)(x+2)},\quad \text{(ii)}\ \frac{x^5}{(x^2+x+1)(x+1)},\quad \text{(iii)}\ \frac{x^4}{(x^2+1)^2(x+1)}.$$

(i)

$$\begin{aligned}
x^5 &= (x^2+3x+2)(x^3-3x^2+7x-15)+31x+30 \\
\Rightarrow \quad \frac{31x+30}{(x+1)(x+2)} &= \frac{A}{x+1}+\frac{B}{x+2} \\
&= \frac{-1}{x+1}+\frac{32}{x+2} \\
\Rightarrow \quad \frac{x^5}{(x+1)(x+2)} &= x^3-3x^2+7x-15-\frac{1}{x+1}+\frac{32}{x+2}
\end{aligned}$$

(ii)

$$\begin{aligned}
x^5 &= (x^3+2x^2+2x+1)(x^2-2x+2)-x^2-2x-2 \\
\Rightarrow \quad \frac{x^2+2x+2}{(x^2+x+1)(x+1)} &= \frac{Ax+B}{x^2+x+1}+\frac{C}{x+1} \\
&= \frac{1}{x^2+x+1}+\frac{1}{x+1} \\
\Rightarrow \quad \frac{x^5}{(x^2+x+1)(x+1)} &= x^2-2x+2-\frac{1}{x^2+x+1}-\frac{1}{x+1}
\end{aligned}$$

(iii)

$$\begin{aligned}
\frac{x^4}{(x^2+1)^2(x+1)} &= \frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}+\frac{E}{x+1} \\
\Rightarrow \quad \frac{(-1)^4}{2^2} &= E \\
\Rightarrow \quad E &= \frac{1}{4} \\
\frac{x^4}{(x^2+1)^2(x+1)}-\frac{1}{4(x+1)} &= \frac{4x^4-(x^2+1)^2}{4(x^2+1)^2(x+1)} \\
&= \frac{4x^4-x^4-2x^2-1}{4(x^2+1)^2(x+1)} \\
&= \frac{(x+1)(3x^3-3x^2+x-1)}{4(x^2+1)^2(x+1)} \\
&= \frac{3x^3-3x^2+x-1}{4(x^2+1)^2} \\
&= \frac{(x^2+1)(3x-3)-2x+2}{4(x^2+1)^2}
\end{aligned}$$

$$\Rightarrow \qquad \frac{x^4}{(x^2+1)^2(x+1)} = \frac{1-x}{2(x^2+1)^2} + \frac{3x-3}{4(x^2+1)} + \frac{1}{4(x+1)}$$

Question (1918 STEP IIII Q206)

Verify by the use of partial fractions or otherwise that

$$\operatorname{cosec}(x-\alpha_1)\operatorname{cosec}(x-\alpha_2) = A_1\cot(x-\alpha_1) + A_2\cot(x-\alpha_2),$$
$$\operatorname{cosec}(x-\alpha_1)\operatorname{cosec}(x-\alpha_2)\operatorname{cosec}(x-\alpha_3) = B_1\operatorname{cosec}(x-\alpha_1) + B_2\operatorname{cosec}(x-\alpha_2) + B_3\operatorname{cosec}(x-\alpha_3)$$

where $A_1 = \operatorname{cosec}(\alpha_1-\alpha_2)$, $B_1 = \operatorname{cosec}(\alpha_1-\alpha_2)\operatorname{cosec}(\alpha_1-\alpha_3)$ with corresponding forms for the other constants. Extend the process to the product of n distinct cosecants, explaining why the sums contain the n cotangents or the n cosecants according as n is even or odd.

Question (1931 STEP IIII Q204)

The polynomials $f(x)$ and $\phi(x)$ are of degrees n and m respectively, n being greater than m. Shew that, if $f(x)$ and $\phi(x)$ have no common factor, it is possible to find polynomials $F(x)$ and $\Phi(x)$ such that

$$f(x)F(x) + \phi(x)\Phi(x) = 1.$$

Shew further that it is always possible to find an F whose degree is less than m, and that the polynomials F and Φ are then unique. Hence prove that, if the polynomials A and B have no common factor, the rational function C/AB can be expressed in only one way as

$$D + \frac{P}{A} + \frac{Q}{B},$$

where P and Q are polynomials whose degrees are less than those of A and B respectively. If A is $(x-a)^n$, and $(x-a)$ is not a factor of C, then P/A can be expressed as

$$\frac{\lambda_1}{x-a} + \frac{\lambda_2}{(x-a)^2} + \cdots + \frac{\lambda_n}{(x-a)^n}.$$

Shew that

$$\lambda_{n-r} = \frac{1}{n!}\frac{d^r}{dx^r}\left[\frac{C(x)}{B(x)}\right]_{x=a}.$$

Question (1939 STEP IIII Q206)

Prove that a real rational function of x may be expressed as the sum of a polynomial and real partial fractions.

Express in this way $\frac{1}{x(x^2+1)^2}$.

Question (1942 STEP I Q301)

Determine λ so that the equation in x

$$\frac{2A}{x+a} + \frac{\lambda}{x} - \frac{2B}{x-a} = 0$$

may have equal roots; and if $\lambda_1, \lambda_2, x_1, x_2$ be the two values of λ and the two corresponding values of x, prove that

$$x_1 x_2 = a^2, \quad \lambda_1 \lambda_2 = (A-B)^2.$$

Question (1915 STEP II Q303)

Put into real partial fractions

(i) $\frac{1}{(x+1)^2(x+2)(x+3)}$,

(ii) $\frac{1}{x^4+1}$.

Question (1924 STEP II Q305)

$P(x), Q(x)$ are given polynomials of which the latter can be expressed as the product of real linear factors. Into what Partial Fractions can the function $P(x)/Q(x)$ be decomposed? Prove your result and shew that such decomposition is possible in one way only.

Question (1931 STEP II Q401)

Express in partial fractions

$$\frac{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}{(x-a)(x-b)(x-c)(x-d)}.$$

Hence or otherwise show that

$$\frac{(a-\alpha)(a-\beta)(a-\gamma)(a-\delta)}{(a-b)(a-c)(a-d)} + \frac{(b-\alpha)(b-\beta)(b-\gamma)(b-\delta)}{(b-a)(b-c)(b-d)} + \text{two similar terms}$$
$$= a+b+c+d-\alpha-\beta-\gamma-\delta.$$

Question (1939 STEP III Q401)

Express

$$\frac{x}{(x-2)^5(x+1)(x-1)}$$

in partial fractions, and verify by taking $x=3$.

$$\frac{x}{(x-2)^5(x+1)(x-1)} = \sum_{i=1}^{5} \frac{A_i}{(x-2)^i} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$\Rightarrow \quad \frac{x}{(x-2)^5(x-1)} = (x+1)\sum_{i=1}^{5}\frac{A_i}{(x-2)^i} + B + \frac{C(x+1)}{x-1}$$

$$x=-1: \quad \frac{-1}{(-3)^5(-2)} = B$$

$$\Rightarrow \quad B = -\frac{1}{2\cdot 3^5}$$

$$\Rightarrow \quad \frac{x}{(x-2)^5(x+1)} = (x-1)\sum_{i=1}^{5}\frac{A_i}{(x-2)^i} + \frac{B(x-1)}{x+1} + C$$

$$x=1: \quad \frac{1}{(-1)^5\cdot 2} = C$$

$$\Rightarrow \quad C = -\frac{1}{2}$$

$$\frac{x}{(x-2)^5(x+1)(x-1)} + \frac{1}{2(x-1)} + \frac{1}{2\cdot 3^5(x+1)} = \frac{x + \frac{1}{2}(x+1)(x-2)^5 + \frac{1}{2\cdot 3^5}(x-1)(x-2)^5}{(x-2)^5(x+1)(x-1)}$$

$$=$$

Question (1933 STEP III Q504)

If $\phi(x)$ is a polynomial of degree not greater than that of a polynomial $f(x)$, shew that

$$\frac{\phi(x)}{(x-a)f(x)} \equiv \frac{\phi(a)}{(x-a)f'(a)} + \frac{\text{a polynomial in } x}{f(x)},$$

provided $f(a) \neq 0$. Discuss the case $f(a) = 0$. Expand $\dfrac{2x^8 - 5x^4 + 2x^3 + 6x^2 - 2}{(x+2)(x^2-1)^3}$ in a series of ascending powers of x, stating carefully the general term.

Question (1923 STEP IIII Q502)

Give a general account of the resolution of a fraction (whose numerator and denominator are polynomials) into partial fractions. Resolve into partial fractions $\dfrac{x^2}{(x+2)^2(x^2+1)}$.

Question (1921 STEP I Q607)

Solve the equations

$$x(y+a) - ay = y(z+a) - az = z(x+a) - ax$$

$$3(x+y+z) = 10a.$$

Resolve into partial fractions

$$\frac{1}{(1-2x)(1-8x^3)}.$$

Question (1925 STEP I Q608)

Express as partial fractions $\dfrac{ay}{(y+a)^2(y-a)}$ and deduce the partial fractions for $\dfrac{x(x^2-1)}{(x^2+x-1)^2(x^2-x-1)}$.

$$
\begin{aligned}
& & \frac{ay}{(y+a)^2(y-a)} &= \frac{A}{y+a} + \frac{B}{(y+a)^2} + \frac{C}{y-a} \\
&\Rightarrow & ay &= A(y+a)(y-a) + B(y-a) + C(y+a)^2 \\
&y = a: & a^2 &= 4Ca^2 \\
&\Rightarrow & C &= \frac{1}{4} \\
&y = -a: & -a^2 &= -2Ba \\
&\Rightarrow & B &= \frac{a}{2} \\
&y = 0: & 0 &= -a^2A - aB + Ca^2 \\
& & 0 &= -a^2A - \frac{a^2}{2} + \frac{a^2}{4} \\
&\Rightarrow & A &= -\frac{1}{4} \\
&\Rightarrow & \frac{ay}{(y+a)^2(y-a)} &= -\frac{1}{4(y+a)} + \frac{a}{2(y+a)^2} + \frac{1}{4(y-a)}
\end{aligned}
$$

If $a = x$, and $y = x^2 - 1$ then

$$
\frac{x(x^2-1)}{(x^2+x-1)^2(x^2-x-1)} = \frac{x}{2(x^2+x-1)^2} + \frac{1}{4(x^2-x-1)} - \frac{1}{4(x^2+x-1)}
$$

Question (1922 STEP II Q601)

Express $\frac{2x^3+x^2+2}{(x^2-1)(x^2+2x+2)}$ as the sum of three partial fractions.

Question (1924 STEP IIII Q603)

Illustrate the methods of expressing the ratio of two rational functions of x as a sum of partial fractions by considering the cases

$$
\text{(i) } \frac{x^4+x^2}{x^6-1}, \quad \text{(ii) } \frac{x^6}{(x-1)^3(x^2+1)^2}.
$$

Question (1914 STEP I Q703)

Express $\dfrac{1}{(1-x)(1-x^2)}$ as the sum of three partial fractions, and shew that the coefficient of x^n in the expansion in ascending powers of x is

$$\frac{1}{2}\left(n+1+\cos^2\frac{n\pi}{2}\right).$$

Shew also that the coefficient of $x^p y^q$ in the expansion of $(x+y+xy)^n$ is

$$\frac{n!}{(n-p)!(n-q)!(p+q-n)!}.$$