Question (1980 STEP I Q3)

Let $p(x)$ be a polynomial of degree 4, with real coefficients, and satisfying the property that, for all rational numbers α, $p(\alpha)$ is a rational number. Prove that $p(x)$ has rational coefficients. If $q(x)$ is a polynomial with rational coefficients, and $q(n)$ is an integer for every integer n, does it follow that $q(x)$ has integer coefficients? Give either a proof or a counter-example. [A rational number is a number of the form p/q where p, q are integers, $q \neq 0$.]

Question (1981 STEP II Q3)

Suppose that $f(n)$ is a polynomial with rational coefficients of degree $k > 0$ in n where n is an integer. Show that the function

$$g(n) = f(n) - f(n-1) \tag{1}$$

is a polynomial of degree $k-1$. Show also that if $f(n)$ has the form $g(n) \cdot g(n+1) \cdot g(n+2)$ (for some polynomial $g(n)$), then $g(n)$ is divisible by $g(n) \cdot g(n+1)$. Evaluate the sum

$$f(n) = \sum_{r=1}^{n} \frac{1}{r^3(r+1)^3(r^2+r+1)} \tag{2}$$

and hence show that $f(n)$ is a perfect cube for all values of n.

Question (1980 STEP III Q3)

Let $x_1, \ldots, x_n$ be distinct real numbers. Write down an expression for a polynomial e_k, of degree $n-1$, such that

$$e_k(x_l) = \begin{cases} 1 & (l = k), \\ 0 & (l \neq k). \end{cases}$$

Given real numbers $\alpha_1, \ldots, \alpha_n$, find a polynomial p, of degree at most $n-1$, for which $p(x_k) = \alpha_k$ $(k = 1, \ldots, n)$. Show further, given numbers $\beta_1, \ldots, \beta_n$, that there is a polynomial q, of degree at most $2n-1$, such that both $q(x_k) = \alpha_k$ and $q'(x_k) = \beta_k$ $(k = 1, \ldots, n)$. [It is sufficient to prove the existence of q; you are not expected to find its coefficients in an explicit form. In the last part of the question, you may find it helpful firstly to find a polynomial η_k such that

$$\eta_k(x_l) = 0 \ (l = 1, \ldots, n), \ \eta_k'(x_k) = 1, \ \eta_k'(x_l) = 0 \ (l \neq k, l = 1, \ldots, n).]$$

Question (1982 STEP III Q5)

Let m and n be integers with $0 \leq m \leq n$. The function $f_{n,m}(x)$, defined for $|x| \neq 1$, is given by

$$f_{n,m}(x) = \begin{cases} \frac{(x^n-1)(x^{n-1}-1)\dots(x^{n-m+1}-1)}{(x^m-1)(x^{m-1}-1)\dots(x-1)} & \text{if } m > 0, \\ 1 & \text{if } m = 0 \end{cases}$$

Prove that for $0 < m < n$,

$$f_{n,m}(x) = f_{n-1,m-1}(x) + x^m f_{n-1,m}(x).$$

Show that $f_{n,m}(x)$ can be expressed as a polynomial of degree $m(n-m)$ for $|x| \neq 1$, and that the value of this polynomial when $x = 1$ is equal to the binomial coefficient $\binom{n}{m}$.

Question (1977 STEP IIII Q1)

Polynomials $C_r(x)$ are defined by

$$C_0(x) = 1,$$

$$C_r(x) = \frac{x(x-1)\dots(x-r+1)}{r!} \text{ for } r \geq 1.$$

(i) Show that if n is an integer, then so is $C_r(n)$. (ii) Show that any polynomial $p(x)$ with rational coefficients can be expressed in the form

$$b_k C_k(x) + b_{k-1} C_{k-1}(x) + \dots + b_0 C_0(x),$$

where all the b_r's are rational and k is the degree of $p(x)$. Suppose further that whenever n is an integer, then so is $p(n)$. Show that all the b_r's are integers. (iii) Suppose that $p(x)$ is a polynomial with real coefficients such that whenever a is a rational number then so is $p(a)$. Show that the coefficients of $p(x)$ are all rational.

Question (1977 STEP IIII Q8)

(i) By considering $A(1+\eta-x^2)^n$ for suitable values of A, η and n, show that, given $\epsilon > 0$ and $0 < \beta < \alpha < 1$, we can find a polynomial $P(x)$ such that

$$P(x) \geq 1 \text{ for } |x| \leq \beta,$$

$$0 \leq P(x) \leq 1 \text{ for } \beta \leq |x| \leq \alpha,$$

$$0 \leq P(x) \leq \epsilon \text{ for } \alpha \leq |x| \leq 1.$$

(ii) Show that, given $\epsilon > 0$, there is a polynomial $Q(x)$ such that

$$|Q(x)| \leq \epsilon \text{ for } -1 \leq x \leq -\epsilon,$$

$$-\epsilon \leq Q(x) \leq 1+\epsilon \text{ for } -\epsilon \leq x \leq \epsilon,$$

$$|Q(x)-1| \leq \epsilon \text{ for } \epsilon \leq x \leq 1.$$

(iii) Show that, given $\epsilon > 0$ and $0 < a \leq 1$, there is a polynomial $R(x)$ such that

$$|R(x)| \leq \epsilon \text{ for } a \leq |x| \leq 1,$$

$$|R(x)-(1-a^{-1}|x|)| \leq \epsilon \text{ for } |x| \leq a.$$

Question (1982 STEP IIII Q6)

Express the sum of the fifth powers of the roots of a cubic equation in terms of the sum of the roots, the sum of the squares of the roots and the product of the roots. Prove that $\frac{(x-y)^5+(y-z)^5+(z-x)^5}{(x-y)^2+(y-z)^2+(z-x)^2} = \frac{5}{2}(x-y)(y-z)(z-x)$ for all distinct real numbers x, y, z.

Question (1959 STEP I Q104)

Prove that, if $h(x)$ is the H.C.F. of two polynomials $p(x)$, $q(x)$, then polynomials $A(x)$, $B(x)$ exist such that

$$A(x)p(x) + B(x)q(x) \equiv h(x).$$

Obtain an identity of this form when

$$p(x) = x^{10}-1, \quad q(x) = x^6-1.$$

Question (1962 STEP I Q104)

$x_1, \ldots, x_n$ are distinct numbers and, for $1 \le r \le n$, $p_r(x)$ is written for

$$(x - x_1) \ldots (x - x_{r-1})(x - x_{r+1}) \ldots (x - x_n).$$

By considering

$$\sum_{r=1}^{n} \alpha_r p_r(x),$$

for suitably chosen α_r, show that it is possible to find a polynomial of degree not exceeding $n - 1$ which takes given values at $x_1, \ldots, x_n$. Similarly, by considering

$$\sum_{r=1}^{n} (\beta_r x + \gamma_r)\{p_r(x)\}^3,$$

show that it is possible to find a polynomial of degree not exceeding $2n - 1$ which takes given values at $x_1, \ldots, x_n$ and whose first derivative also takes given values at these points.

Question (1962 STEP II Q102)

Explain how turning values and points of inflexion of the function $y = f(x)$ can be found by studying the successive derivatives of y. Find the values of x for which the function

$$y = \frac{x^3 - x^2 + 4}{x^3 + x^2 + 4}$$

has turning values and discuss their character. How many real roots has the equation

$$x^3(a - 1) + x^2(a + 1) + 4(a - 1) = 0$$

for different values of a?

Question (1950 STEP I Q102)

Explain briefly how to find the H.C.F. of two integers or two polynomials. If m and n are positive integers whose H.C.F. is k, prove that the H.C.F. of the integers $2^m - 1$ and $2^n - 1$ is $2^k - 1$ and that the H.C.F. of the polynomials $x^{2^m} - x$ and $x^{2^n} - x$ is $x^{2^k} - x$.

Question (1952 STEP I Q102)

Prove what you can about the number of real roots of each of the equations

(i) $(x - a_1)(x - a_2) \ldots (x - a_n) + (x - b_1)(x - b_2) \ldots (x - b_n) = 0$, where

$$a_1 > b_1 > a_2 > b_2 \cdots > a_n > b_n;$$

(ii) $(x - a)^m + (x - b)^m = 0$ where $a > b$ and m is a positive integer.

Question (1950 STEP IIII Q201)

Find whether any of the roots of the equation

$$x^5 + 8x^4 + 6x^3 - 42x^2 - 19x - 2 = 0$$

are integers, and solve it completely.

Question (1950 STEP II Q405)

Show that the conditions that an algebraic equation $f(x) = 0$ has a double root at $x = a$ are that $f(a) = f'(a) = 0$. If the equation

$$x^4 - (a+b)x^3 + (a-b)x - 1 = 0$$

has a double root, prove that

$$a^{\frac{2}{3}} - b^{\frac{2}{3}} = 2^{\frac{2}{3}}.$$

Question (1951 STEP II Q408)

Prove that the equation $x^3 - 3px^2 + 4q = 0$ will have three real roots if p and q are the same sign and $p^6 > q^2$. Show that two roots will be positive or negative as the sign of p and q is positive or negative. (It may be assumed that neither p nor q vanish.) Find out by these results as much as possible about the roots of the equation

$$x^3 - 6x^2 + 16 = 0.$$

Question (1950 STEP II Q201)

Find a polynomial of the ninth degree $f(x)$, such that $(x-1)^5$ divides $f(x) - 1$ and $(x+1)^5$ divides $f(x) + 1$. Prove that the quotients do not vanish for any real value of x.

Question (1948 STEP IIII Q302)

Find the highest common factor of

$$f(x) = 27x^4 + 27x^3 + 22x + 4 \quad \text{and} \quad g(x) = 54x^3 + 27x + 11.$$

Hence show that the equation $f(x) = 0$ has a repeated root, and solve it completely.

Question (1948 STEP II Q401)

Solve the equation:

$$8x^6 - 54x^5 + 109x^4 - 108x^3 + 109x^2 - 54x + 8 = 0.$$

Solve the equation:

$$8x^6 - 54x^5 + 109x^4 - 108x^3 + 109x^2 - 54x + 8 = 0.$$

$$
\begin{aligned}
&& 0 &= 8x^6 - 54x^5 + 109x^4 - 108x^3 + 109x^2 - 54x + 8 \\
\Rightarrow && 0 &= 8(x^3 + x^{-3}) - 54(x^2 + x^{-2}) + 109(x + x^{-1}) - 108 \\
&& &= 8((x + x^{-1})^3 - 3(x + x^{-1})) - 54((x + x^{-1})^2 - 2) + 109(x + x^{-1}) - 108 \\
&& &= 8z^3 - 24z - 54z^2 + 108 + 109z - 108 \\
&& &= 8z^3 - 54z^2 + 85z \\
&& &= z(8z^2 - 54z + 85) \\
&& &= z(2z - 5)(4z - 17) \\
\Rightarrow && 0 &= x + \frac{1}{x} && \text{(no soln)} \\
&& \frac{5}{2} &= x + \frac{1}{x} \\
\Rightarrow && 0 &= 2x^2 - 5x + 2 \\
&& &= (2x - 1)(x - 2) \\
\Rightarrow && x &= 2, \frac{1}{2} \\
&& \frac{17}{4} &= x + \frac{1}{x} \\
\Rightarrow && 0 &= 4x^2 - 17x + 4 \\
&& &= (4x - 1)(x - 4) \\
&& x &= 4, \frac{1}{4}
\end{aligned}
$$

Therefore $x \in \left\{2, \frac{1}{2}, 4, \frac{1}{4}\right\}$

Question (1945 STEP II Q202)

Prove that, if the equation

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

is satisfied for more than n distinct values of x, then $a_0, a_1, \ldots, a_n$ are all zero. A function $f(x)$ is said to be rational, if it can be expressed in the form $P(x)/Q(x)$, where $P(x), Q(x)$ are polynomials in x. A function $f(x)$ is said to be periodic, with period k, if $f(x + k) = f(x)$ for all values of x for which $f(x)$ is defined. Prove that a periodic function cannot be a rational function.

Question (1947 STEP II Q201)

If P and Q are polynomials and if the degree of Q is less than the degree of P, show that polynomials $P_0, P_1, P_2, \ldots$ all of degree less than Q can be found such that

$$P = \Sigma P_i Q^i.$$

Prove that the polynomials P_i are unique.
If the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the equation $Q = 0$ are all different, find the polynomial of least degree which takes the value a whenever $Q = 0$ and whose derived polynomial takes the value b whenever $Q = 0$.
[The derived polynomial of $P(x)$ is the coefficient of h in the expansion of $P(x+h)$ in powers of h.]

Question (1947 STEP II Q202)

State a necessary and sufficient condition that

$$z^2 + 4axz + 6byz + 4cxy + dy^2 + 2\lambda(x^2 - yz)$$

shall be the product of two linear factors.
By taking $z = x^2, y = 1$, state briefly the steps to be taken in order to find the roots of the quartic equation

$$x^4 + 4ax^3 + 6bx^2 + 4cx + d = 0.$$

Hence find the roots of the equation

$$x^4 - x^3 - 4x^2 + x + 1 = 0.$$

Question (1948 STEP II Q301)

A number of the form p/q, where p and q are integers ($q \neq 0$), is said to be rational. Prove that a rational root of the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,$$

where $a_1, a_2, \ldots, a_n$ are integers, is necessarily an integer. Prove further that, if $|a_n|$ is a prime number, the equation cannot have more than three distinct rational roots.
Find the rational roots of the equation

$$8x^5 - 4x^4 - 2x^3 - 3x^2 + 1 = 0,$$

and hence solve the equation completely.

Question (1914 STEP I Q106)

Prove that, if n is a prime number,

(i) the coefficients in $(1+x)^n$ except the first and last are all divisible by n.

(ii) the successive coefficients in $(1+x)^{n-1}$ when divided by n leave remainders 1 and $n-1$ alternately.

(iii) the successive coefficients in $(1+x)^{n-2}$ when divided by n leave remainders $1, n-2, 3, n-4, 5, n-6, \ldots$.

Question (1932 STEP I Q101)

Shew that, if x^4+ax+b has a factor x^2+px+q, then

$$p^6-4bp^2-a^2=0 \quad \text{and} \quad q^6-bq^4-a^2q^3-b^2q^2+b^3=0.$$

Solve the equation

$$x(x-1)(x-2)(x-3)=a(a-1)(a-2)(a-3),$$

and find for what values of a the roots are all real.

Question (1941 STEP I Q106)

Prove that in general three normals (real or imaginary) can be drawn to a parabola from an arbitrary point in its plane. If M is a fixed point on the parabola and P a variable point on the normal at M, show that the line joining the feet of the other two normals from P is parallel to a fixed direction.

Question (1914 STEP II Q209)

A quadratic function of x takes the values y_1, y_2, y_3 corresponding to three equidistant values of x; prove that, if $y_1+y_3>2y_2$, the minimum value of the function is

$$y_2-\frac{(y_3-y_1)^2}{8(y_1+y_3-2y_2)}.$$

Question (1924 STEP II Q208)

Find values of a, b, c, d such that the curve $y=ax^3+bx^2+cx+d$ touches the lines $3x-y-6=0, 3x+3y+2=0$ at their points of intersection with the axes of x and y respectively. Prove that the curve touches the axis of x, and that the curvature at the point of contact is 2.

Question (1933 STEP IIII Q203)

$g(x), h(x)$ are given polynomials, of degrees m, n respectively ($m \geq n$). Prove that the degree of a polynomial (not vanishing identically) which can be written in the form

$$(1) \qquad G(x)g(x) + H(x)h(x),$$

where $G(x)$ and $H(x)$ are polynomials, can be as small as, but not smaller than, a definite integer $\nu(\geq 0)$. Prove also that the polynomial $\chi(x)$, which is of the form (1) and of degree ν, and in which the coefficient of x^ν is unity, is unique. Prove that $\chi(x)$ is a common factor of $g(x)$ and $h(x)$. Prove also that in the expression of $\chi(x)$ in the form (1), $G(x)$ and $H(x)$ can be found of degrees less than n and m respectively.

Question (1915 STEP IIIII Q205)

Resolve

$$12x^2 + x - 35 \quad \text{and} \quad bc - ca - ab + a^2$$

each into two factors and $(6x^2 - x)^2 - 7(6x^2 - x) + 10$ into four factors.

*10. Solve the equations:

(1) $(a - b)x - ay = b(2a - 3b)$; $(x - y) = 6b$.

(2) $3x + 5y = x^2 - 2y^2 = 1$.

*11. When eggs are lowered $1\frac{1}{2}$d. per dozen, one dozen more can be bought for 30s. Find the price per dozen.

Question (1938 STEP I Q301)

Find the highest common factor of the two polynomials

$$f(x) = x^4 - 13x^3 + 58x^2 - 96x + 36$$
$$g(x) = x^4 - 11x^3 + 36x^2 - 24x - 36.$$

Find all the roots of the equation $f^2(x) - g^2(x) = 0$.

Question (1939 STEP I Q302)

The quartic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

has four real roots. Prove that

(i) $8b < 3a^2$;

(ii) $8bd < 3c^2$;

(iii) $8bd = 3c^2$, if $8b = 3a^2$;

(iv) $c = d = 0$, if $8b < 3a^2$, $8bd = 3c^2$.

Question (1923 STEP IIII Q302)

Having given that a quadratic function of x assumes the values V_1, V_2, V_3 for the values $x = a, x = b, x = c$ prove that the function must be

$$V_1\frac{(x-b)(x-c)}{(a-b)(a-c)} + V_2\frac{(x-c)(x-a)}{(b-c)(b-a)} + V_3\frac{(x-a)(x-b)}{(c-a)(c-b)}.$$

A variable quantity which can be represented by a quadratic function of the time assumes the values 144, 15.6, 18 at 10 a.m., 1 p.m. and 2 p.m. on a certain day. Find the value at noon, and find at what time the quantity assumes its least value.

Question (1926 STEP I Q401)

Solve the equations:

(i) $16x(x+1)(x+2)(x+3) = 9$,

(ii)

$$x + y + z = axyz,$$
$$yz + zx + xy = -b,$$
$$(1+x^2)(1+y^2)(1+z^2) = (1+b)^2,$$

where a is not equal to 1.

Question (1927 STEP I Q401) (i) Solve the equation

$$\frac{4}{x^2-2x} - \frac{2}{x^2-x} = x^2 - x.$$

(ii) Prove that $x = \dfrac{2}{n+1}$ satisfies the equation

$$(x-1)^3 + (2x-1)^3 + (3x-1)^3 + \cdots + (nx-1)^3 = 0,$$

and find the quadratic equation satisfied by the other two roots.

Question (1941 STEP III Q408)

If $y = \frac{x^4+x^2-12}{x^4-4}$, determine the range of values possible for y when x is real. Sketch the graph of the function.

Question (1914 STEP IIII Q401)

Factorize

(1) $(b-c)^5 + (c-a)^5 + (a-b)^5$.

(2) $(x+y+z)^7 - x^7 - y^7 - z^7$.

Solve the equations

$$yz + zx + xy = a^2 - x^2 = b^2 - y^2 = c^2 - z^2.$$

Question (1914 STEP IIII Q405)

Determine graphically or otherwise for what values of λ the equation $2x^3 - 15x^2 + 24x - \lambda = 0$ has three real roots.

Question (1925 STEP II Q505)

Investigate the maxima and minima of the function $(x+1)^5/(x^5+1)$ and trace its graph. Prove that the equation $(x+1)^5 = m(x^5+1)$ has three real roots if $0 < m < 16$ and only one real root (-1) if $m < 0$ or $m > 16$.

Question (1927 STEP II Q501)

Find the linear factors of

$$a^3(b-c) + b^3(c-a) + c^3(a-b).$$

Show that if $x^3 + y^3 + z^3 = 3mxyz$, and

$$ax^2 + by^2 + cz^2 = 0$$
$$ayz + bzx + cxy = 0$$

then

$$a^3 + b^3 + c^3 = 3mabc.$$

Question (1934 STEP III Q506)

The ordinate of any point on a curve is equal to a cubic polynomial in the abscissa. The curve touches Ox at the origin and intersects that axis again at the point $(2,0)$. Prove that the tangent to the curve at its point of inflexion cuts Ox at the point $(\frac{2}{3}, 0)$.

Sketch the curve.

Question (1915 STEP IIII Q501)

Solve the equation
$$(x-1)(x+2)(x+3)(x+6) = 160.$$
Eliminate x, y, z from
$$x+y-z=a, \quad x^2+y^2-z^2=b^2, \quad x^3+y^3-z^3=c^3, \quad xyz=d^3.$$

Question (1914 STEP I Q702)

Find the factors of
$$a^3(b-c)+b^3(c-a)+c^3(a-b).$$
Shew that if
$$x^3+y^3+z^3=3xyz,$$
$$ax^2+by^2+cz^2=0,$$
$$ayz+bzx+cxy=0,$$
then
$$a^3+b^3+c^3=3abc.$$

Question (1924 STEP I Q707)

Prove that $a+b-c-d$ is a factor of
$$(a+b+c+d)^3-6(a+b+c+d)(a^2+b^2+c^2+d^2)+8(a^3+b^3+c^3+d^3);$$
and resolve this expression into its simple factors.

Question (1922 STEP I Q813)

Let $f_n(x)$ be a polynomial defined by the equations
$$f_0(x)=1, f_1(x)=x, f_n(x)=(a_nx+b_n)f_{n-1}(x)-c_nf_{n-2}(x), \quad (n=2,3,\dots)$$
where a_n, b_n, c_n are functions of n and $a_n>0, c_n>0$. Prove that all the roots of $f_n(x)=0$ are real and that between any two consecutive roots of $f_n(x)=0$ there lies a root of $f_{n-1}(x)=0$. Defining Legendre's nth polynomial $P_n(x)$ as the coefficient of h^n in the expansion of
$$\frac{1}{\sqrt{1-2xh+h^2}}$$
prove that all the roots of $P_n(x)=0$ are real.

Question (1914 STEP II Q802)

Express the left-hand side of the equation

$$x^4 + 8x^3 - 12x^2 + 104x - 20 = 0$$

as the product of two quadratics with rational coefficients, and solve the equation.