Question (1979 STEP I Q1)

Sketch the graph of the function given by

$$f(x) = \frac{x-a}{x(x-2)},$$

where a is a constant, in each of the following cases:

1. $0 < a < 2$,
2. $a > 2$.

Explain briefly how you deduce the salient features of these graphs.

None

Question (1982 STEP I Q9)

Find a, b such that the function $f(x) = \frac{(ax+b)}{(x-1)(x-4)}$ has a stationary value at $x = 2$ with $f(2) = -1$. Show that $f(x)$ has a maximum at $x = 2$, and sketch the curve.

Question (1973 STEP II Q1)

Sketch the graph of $z(t) = (\log t)/t$ in $t > 0$. Find the maximum value of $z(t)$ in this range. How many positive values of t correspond to a given value of z? Hence find how many positive values of y satisfy $x^y = y^x$ for a given positive value of x. Sketch the graph of $x^y = y^x$ in $x > 0$, $y > 0$.

Question (1974 STEP III Q6)

The cubic curve C in the (x, y)-plane is defined by $y^2 = x^3 - x$. Sketch the curve. Let P be the point $(1, 0)$, and let Q be a point (x_0, y_0), lying on C and distinct from P. Show that the line PQ touches C at Q if and only if $x_0^2 - 2x_0 - 1 = 0$. Let P' be the point $(1 - \epsilon, 0)$, where ϵ is small and positive. Sketch all the tangents from P' to C.

Question (1975 STEP III Q8)

Sketch the curve $y^2 = x^3(1 - x^2)$. From your sketch, estimate the number of times the line $y = ax$ cuts the curve for various values of the constant a. Find the range of values of a for which the line $y = ax$ cuts the curve in exactly one point other than the origin.

Question (1968 STEP IIII Q7)

Sketch the curve whose equation is

$$y^2(1 + x^2) = x^2(1 - x^2),$$

and find the area of a loop of the curve.

Question (1965 STEP I Q8)

The end A of a line segment AB of length $2a$ lies on the circle $x^2 + y^2 = a^2$, and B lies on the line $y = 0$. Show that the locus of the mid-point P of AB is the curve

$$(x^2 + y^2)(x^2 + 9y^2) = 4a^2x^2.$$

Sketch this curve, indicating the relation between the position of B on the line $y = 0$ and the position of P on the curve.

Question (1960 STEP I Q101)

Sketch the curve $x^2 = (y - k)^2(y - 2k)$, where x, y are real variables and k is constant, in the three cases (i) $k < 0$, (ii) $k = 0$, (iii) $k > 0$. Describe the nature of the singularity in each case.

Question (1964 STEP I Q209)

Sketch the three curves

$$xy^2 = (a - x)^2(1 - x)$$

for the following three values of the parameter a:

$$a = \frac{1}{2}, 1, 2.$$

Question (1960 STEP I Q310)

Show that the cubic curve whose equation in rectangular Cartesian co-ordinates is

$$x^3 - x^2y - 2xy^2 + 5xy + 2y^2 = 0$$

has a double point at the origin. Find the equations of the asymptotes and the co-ordinates of their finite points of intersection with the curve. Give a sketch of the curve.

Question (1961 STEP I Q310)

Describe the curve

$$(x^2 + y^2)^2 - 4x^2 = a \qquad (1)$$

for $a = -6, -4, -2, 0, 2, 4$. (Accurate diagrams are not necessary.)

Question (1962 STEP I Q307)

Sketch the curve $x^4 + y^4 - 2x^2a = 0$ for the values 2, 1, $\frac{1}{4}$, 0, -1 of the parameter a. A tetrahedron has the property that any two opposite edges are perpendicular. Prove that the line joining this point to the mid-point of any edge of the tetrahedron is equal and parallel to the line joining the mid-points of the opposite edge to the circumcentre of the tetrahedron.

Question (1960 STEP IIII Q206)

Sketch the curves

$$x^n + y^n = 1$$

for $n = -1, 1, 2, 3, 4$. Also, sketch the curves $y = f(x)$, $y = f'(x)$, for a function $f(x)$ which obeys

$$f'(0) < 0, \quad f''(x) > 0;$$

$$\frac{f(x)}{x} \to 1 \text{ as } x \to +\infty$$

in the range $x > 0$.

Question (1963 STEP IIII Q207)

Prove that, if no two of the real numbers $a_1, a_2, \ldots, a_n$ are equal, and all the real numbers $A_1, A_2, \ldots, A_n$ are positive, then the equation

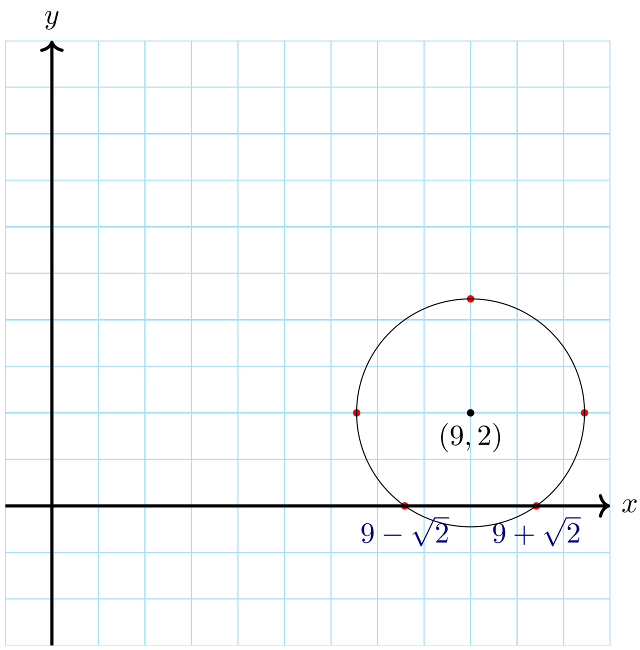
$$\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots + \frac{A_n}{x-a_n} = 0$$

has exactly $n-1$ real roots.

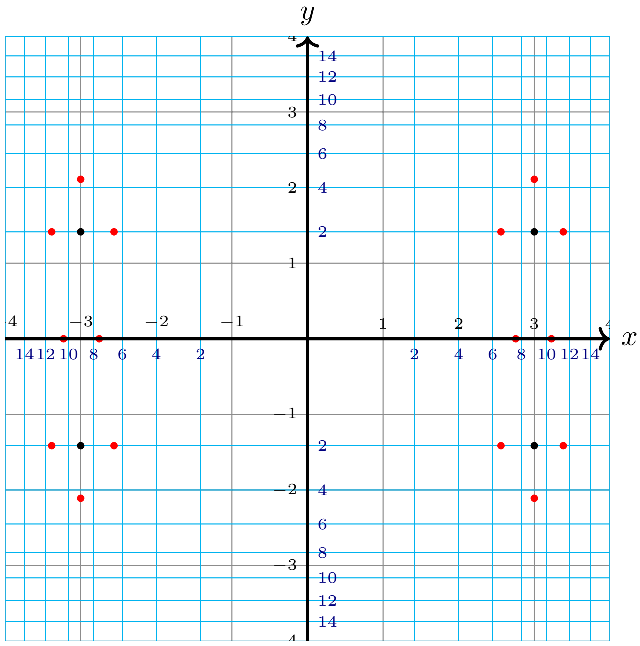
Question (1958 STEP IIII Q306)

Sketch the curve $(x^2-9)^2 + (y^2-2)^2 = 6$.

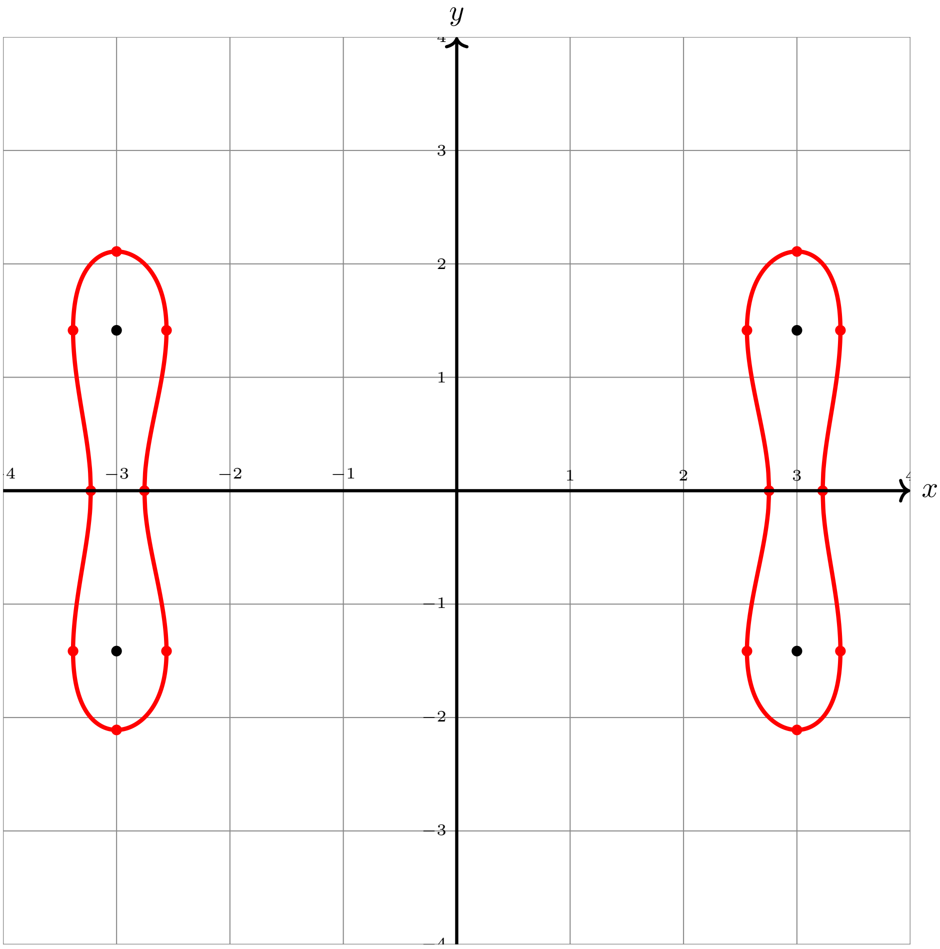
It should be clear by now what transformation we are going to use: $X = x^2, Y = y^2$, so first we will sketch $(X-9)^2 + (Y-2)^2 = 6$



Notice our grid is going to get transformed in *both* directions, so we will just draw our grid first as well as adding a few key points (turning points, intersection of axes, and the centre of the circle).



$$(x^2 - 9)^2 + (y^2 - 2)^2 = 6$$



Notice how the lines are smooth (and vertical) as the cross the x-axis.

Question (1959 STEP IIII Q309)

Sketch the cubic curve

$$(xy - 12)(x + y - 9) = a$$

(i) for a small positive value of the constant a, and

(ii) for a small negative value of the constant a.

For what value of a does the curve have an isolated point?

Question (1958 STEP II Q104)

Sketch the curve

$$x^3 + y^2 = 3xy.$$

By rotating the axes through 45°, or otherwise, find the area of its loop.

Question (1964 STEP II Q103)

Prove that the curve given by $x^y = y^x$ in the region $x > 0$, $y > 0$ of the Cartesian plane has just two branches, and sketch them. What are the coordinates of the point where they cross?

Question (1959 STEP II Q409)

Sketch roughly the possible forms of the curve given by the equation

$$y(ax^2 + 2bx + c) = a'x^2 + 2b'x + c',$$

where a, b, c, a', b', and c' are real, and a and a' are non-zero. Prove that a necessary condition for y to take every real value at least once as x takes all real values can be put in the form

$$(ca' - ac')^2 \leqslant 4(ab' - ba')(bc' - cb').$$

Question (1960 STEP II Q202)

Sketch the curve

$$y = \frac{(x-2)(x-3)}{(x-1)(x-4)}.$$

Prove that

$$\frac{dy}{dx} = \frac{-2(2x-5)}{(x-1)^2(x-4)^2}, \quad \frac{d^2y}{dx^2} = \frac{12(x^2-5x+7)}{(x-1)^3(x-4)^3},$$

and deduce that the radius of curvature at the point where the curve is parallel to the x-axis has the value $3^4/4^3$. Find all the points both of whose coordinates x, y are integers, positive, negative or zero.

Question (1950 STEP IIII Q209)

Find the asymptotes of the curve

$$x^3 + 2x^2 + x = xy^2 - 2xy + y.$$

Show that there are values which $y - x$ never takes, and sketch the curve.

Question (1951 STEP IIII Q209)

Sketch roughly the curve

$$y^2(a^2 + x^2) = x^2(a^2 - x^2),$$

and find the area of one of its loops.

Question (1957 STEP IIII Q206)

Sketch the graph of a function $f(x)$ that satisfies the conditions (i) $f(0) = 0$, (ii) $f'(0) < 0$, (iii) $f''(x) > 0$ for $x > 0$, (iv) $f(x)$ tends to a limit as $x \to \infty$. Also sketch the graph of a function $g(x)$ that satisfies the conditions (i) $g(0) = 0$, (ii) $g'(0) < 0$, (iii) $g''(0) > 0$, (iv) $\dfrac{g(x)}{x} \to 1$ as $x \to \infty$.

Question (1955 STEP II Q110)

Sketch the curve

$$(y^2 - 1)^2 - x^2(2x + 3) = 0.$$

Question (1944 STEP IIII Q305)

A family of curves is given by the equation

$$\left(y + \frac{1}{x^3}\right)(3x - 1) = 8\lambda,$$

where λ is a variable parameter which takes positive values only, and $x > \frac{1}{3}$. Show that if $0 < \lambda < 1$, the curves have one real maximum and one real minimum, while if $\lambda > 1$ the curves have no real maximum or minimum. Show also that the locus of the maxima and minima is $yx^3 = 3x - 2$, and that this locus touches the curve $\lambda = 1$ at the point $x = 1, y = 1$.

Question (1944 STEP IIII Q310)

Trace the curve $y^2 = \frac{x^2(3-x)}{1+x}$, and find the area of the loop.

Question (1945 STEP II Q107)

Sketch the curve
$$a(x^2 - y^2) = y^3 \quad (a > 0).$$
Find (i) the position of the centre of curvature of either branch of the curve at the origin, and (ii) the area of the loop.

Question (1946 STEP II Q102)

Prove that, if k is real and $|k| < 1$, the function $\cot x + k \operatorname{cosec} x$ takes all values as x varies through real values. Prove that, if $|k| > 1$, the function takes all values except those included in an interval of length $2\sqrt{(k^2 - 1)}$. Give rough sketches of the graph of
$$y = \cot x + k \operatorname{cosec} x$$
for $-\pi < x < \pi$, in the cases (i) $0 < k < 1$, (ii) $k > 1$.

Question (1948 STEP II Q101)

Sketch the curves $x^n + y^n = 1$, for $n = 10, 11$, and $1/11$.

Question (1946 STEP II Q408)

Trace the curve $(x^2 + y^2)^2 = 8axy^2$, and find the areas of its loops. Show that the smallest circle that will completely circumscribe the curve has radius $3\sqrt{3}/2a$, and find the coordinates of its centre.

Question (1946 STEP II Q201)

Sketch the curve $y = 3x^5 - 5ax^3$ for positive and negative values of the real number a, and hence determine the number and signs of the real roots of the equation
$$3x^5 - 5ax^3 + b = 0,$$
where b is real, in the various cases that may arise.

Question (1946 STEP III Q102)

A number of particles, all of the same weight, are attached to a light string at points P_0, P_1, P_2, ... If the horizontal intervals between these points are all equal, prove that the tensions in P_0P_1, P_1P_2, ..., are proportional to their lengths. Show that P_0, P_1, P_2, ..., lie on a parabola with a vertical axis.

Question (1946 STEP III Q404)

Show that with a suitable choice of axes the equation of the curve in which a uniform flexible chain hangs in equilibrium under gravity can be put in the form

$$y = c \cosh x/c.$$

What is the relation between x and the arc length s from the lowest point? A uniform chain of length l hangs between two points whose distance apart, d, has horizontal and vertical components h and k. Prove that the parameter c of the catenary in which it hangs satisfies the equation

$$2c \sinh(h/2c) = \sqrt{l^2 - k^2}.$$

If l exceeds d by only a small amount, show that c is large and approximately equal to

$$h^2/2\sqrt{3(l^2 - d^2)}.$$

Question (1915 STEP I Q103)

The middle point of a rod AB moves uniformly with given velocity in a circle, centre O, and the end A moves in a straight line through O in the plane of the circle; shew graphically the velocity of B.

Question (1918 STEP I Q108)

Explain in general how to draw the curve showing, on an angle base, the turning moment on the crank shaft of a given engine from given indicator diagrams, assuming the load on the engine constant. The turning moment on an engine running at 120 revs. per min. increases uniformly with reference to the angle from zero to 50,000 lbs. ft., and then decreases uniformly to zero at 180°, and is repeated for the second half of the revolution. Determine approximately the moment of inertia of the fly-wheel to procure that the greatest variation of speed is not more than one per cent. above and below the mean speed; the load being constant.

Question (1918 STEP I Q110)

The following table gives the volume (v) of one pound of dry saturated steam at different pressures (p):

Pressure, lbs. per sq. in.	20	50	80	110	140	170
Volume, cu. ft.	8.52	5.49	4.07	3.24	2.70	

Show that these values approximately fit an equation of the form $pv^n =$ constant, and find the value of n.

Question (1921 STEP I Q109)

Draw the graph of

$$y = \frac{(x-a)(x-4a)}{x-5a}.$$

Find the maximum and minimum values of y.

Question (1922 STEP I Q107)

Find graphically the positive root of

$$x = 2\sin x$$

in which the angle x is measured in radians. Prove that the number of real roots of the equation $x = 10\sin x$ is 7.

Question (1923 STEP I Q106)

Trace the curves given by

$$\sin x = 2\cos y,$$

for which $y = \frac{1}{6}\pi$, and $y = \frac{1}{3}\pi$ when $x = 0$, respectively. Find by means of your graph the solutions (correct to about one-tenth of a radian) of the equations

$$\sin x = 2\cos y, \quad 7x = 4y,$$

for which

$$0 < x < \pi.$$

Question (1925 STEP I Q110)

Find the stationary values of

$$y = 10\frac{x^2+3x}{2x^2+13x-7}.$$

Give a rough sketch of the graph of this equation.

Question (1926 STEP I Q112)

Find the equation of the tangent at $(1, 2)$ to the curve given by

$$xy(x+y) = x^2 + y^2 + 1,$$

and determine the point at which it intersects the curve. Find the asymptotes and trace the curve.

Question (1927 STEP I Q108)

From H a fixed point on a parabola chords HP, HQ are drawn perpendicular to each other. Shew that the locus of the intersection of tangents to the parabola at P and Q is a straight line.

Question (1927 STEP I Q112)

Trace the curve given by $ax^2y = x^2 + y + 1$.

Question (1929 STEP I Q111)

Determine the asymptotes of the curve

$$(y-1)^2(y^2-4x^2) = 3xy.$$

Investigate on which sides of the asymptotes the corresponding branches of the curve lie and trace the curve.

Question (1931 STEP I Q110)

Draw the curves

1. $(a-x)y^2 - (a+x)x^2 = 0$,
2. $xy^2 - (2a-x)(a-x)^2 = 0$,

using the same axes for both curves, and prove that the area of the curvilinear quadrilateral with vertices at their intersections and nodes is $\frac{1}{3}a^2(2\pi + 24 - 15\sqrt{3})$. What volumes will be generated by rotating this area about the axes of x and y respectively?

Question (1932 STEP I Q108)

Find the coordinates of the node of the curve

$$(x+y+1)y + (x+y+1)^2 + y^3 = 0,$$

and the area of the loop at the node.

Question (1933 STEP I Q110)

Draw a sketch of the curve

$$y^2\frac{a^2-x^2}{c^2} = \frac{x^2}{b^2-x^2},$$

where a, b, c are positive and $a < b$. Find the volumes of the solids obtained by the revolution of the loop about (i) the x-axis and (ii) the y-axis.

Question (1934 STEP I Q108)

Trace the curve
$$y^4 - 4axy^2 + 3a^2x^2 - x^4 = 0,$$
and shew that it has tangents parallel to the axis of x at two points for which $x^4 = \frac{3}{4}a^4$.

Question (1936 STEP I Q107)

Sketch the curve
$$y^2 = \frac{2x-1}{x^2-1}.$$
Shew that $x + y = 1$ is an inflexional tangent. Are there any others?

Question (1938 STEP I Q109)

Prove that the maxima of the curve $y = e^{-kx} \sin px$ (k and p being positive constants) all lie on a curve whose equation is $y = Ae^{-kx}$, and find A in terms of k and p. Draw in the same diagram rough sketches of the curves $y = e^{-kx}$, $y = -e^{-kx}$ and $y = e^{-kx} \sin px$ for positive values of x.

Question (1939 STEP I Q108)

Sketch the graph of the function
$$y = e^{-a(x+b/x^2)},$$
where a and b are both positive. Prove that there are always at least two points of inflexion. Find the abscissae of the points of inflexion when $a = 10/27$, $b = 5$.

Question (1940 STEP I Q106)

Find the greatest and the least values of the function
$$\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}$$
(i) for all real values of x, and (ii) for $0 \leq x \leq \pi$.

Question (1913 STEP I Q106)

Sketch the curve $y = \dfrac{x}{(x+1)(x+2)}$ and determine the maximum and minimum values of its ordinate.

Question (1915 STEP I Q114)

Find the maxima and minima of the function

$$y = \sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x.$$

Draw a graph of the function.

Question (1916 STEP I Q110)

Find the asymptotes of the curve

$$x^2(x+y) = x + 4y,$$

and trace the curve.

Question (1917 STEP I Q112)

Shew that the function

$$-4c + 4c^2 + 16c^3 - 16c^4,$$

where $c = \cos\theta$, has maximum values equal to 1 when θ is equal to

$$-\frac{3\pi}{5}, -\frac{\pi}{5}, \frac{\pi}{5}, \text{ and } \frac{3\pi}{5};$$

and draw a graph of the function for values of θ between $-\pi$ and π.

Question (1918 STEP I Q114)

Trace the curve $y^2(a+x) = x^2(3a-x)$, and shew that the area of the loop and the area included between the curve and the asymptote are both equal to $3\sqrt{3}a^2$.

Question (1921 STEP I Q108)

Show that the curve

$$x^2(x+y) - y^2 = 0$$

has a cusp at the origin and the rectilinear asymptote $x + y = 1$, that no part of the curve lies between $x = 0$ and $x = -4$, and that it consists of two infinite branches, one in the second quadrant and the other in the first and fourth quadrants. Give a sketch of the curve.

Question (1922 STEP I Q111)

Find the asymptotes of the curve

$$x(x+y)^2 = 2(5x - 3y),$$

and trace the curve.

Question (1923 STEP I Q112)
Find the asymptotes of the curve

$$(x + y - 1)^3 = x^3 + y^3,$$

and show that they meet the curve only at infinity. Show also that there is no point on the curve for which x lies between a and 1, where a is the real root of the equation

$$3a^3 + 3a^2 - 3a + 1 = 0.$$

Give a sketch of the curve showing its general form.

Question (1924 STEP I Q111)
Prove that the curve

$$2x^2y^2 + x^3 - y^3 - 2xy = 0$$

has (1) a double-point at the origin, each branch having a point of inflexion there, (2) an inflexion at each of the four points $x = \pm 1, y = \pm 1$, the tangents being parallel to the axes, (3) no rectilinear asymptotes. Give a sketch of the curve.

Question (1928 STEP I Q111)
Find the asymptotes of the curve

$$x^4 + 3x^2y + 2x^2y^2 + 2xy + 3x + y = 0;$$

determine on which side or sides the curve approaches each asymptote, and where it cuts the asymptotes.

Question (1914 STEP I Q113)
Trace the curve

$$y = x \pm \sqrt{\{x(x-1)(2-x)\}}.$$

Question (1919 STEP I Q112)
Discuss the maxima and minima of $\tan 3x \cot 2x$, and sketch the general shape of the graph of the function between $x = \pm\frac{1}{2}\pi$.

Question (1920 STEP I Q109)
A curve is given by the equations

$$x = t^3, \quad y = t(t^2 - 5),$$

t being a variable parameter. Give a sketch of the curve, showing its general form. Show in particular that there is a double point and find the tangents there. Prove that the curvature never changes sign.

Question (1920 STEP I Q112)

Prove that the arc S of the evolute of a given curve satisfies in general the equation

$$S = \rho + c,$$

where ρ is the radius of curvature of the given curve at the corresponding point and c is a constant. Determine the total length of the evolute of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Question (1930 STEP I Q105)

Shew that if a uniform heavy string has its ends fixed and hangs freely

$$y = c\cosh\frac{x}{c}, \quad s = c\sinh\frac{x}{c}.$$

ACB is a telegraph wire, the straight line AB being horizontal and of length $2l$, and C the middle point of the wire is at a distance h below AB. Shew that the length of the wire is approximately

$$2l + \frac{4}{3}\frac{h^2}{l} - \frac{28}{45}\frac{h^4}{l^3}.$$

Question (1914 STEP I Q105)

Discuss generally the question of the existence of maxima or minima of the function

$$y = \frac{x^2 + 2ax + b}{x^2 + 2Ax + B},$$

and sketch the various possible forms which the graph of the function may have for different values of the constant coefficients. Illustrate your remarks by reference to the functions

$$\frac{x^2}{x^2 + x + 1}, \quad \frac{x^2 - 1}{x^2 - 4}, \quad \frac{x(x-2)}{(x-1)(x-3)}.$$

Question (1922 STEP I Q104)

Investigate the possible forms of the graph

$$y = \frac{x + a}{x^2 + b},$$

for different values, positive and negative, of a and b.

Question (1927 STEP I Q105)

In connection with the tracing of an algebraic curve $f(x, y) = 0$ explain

1. how to find the approximate form of the curve in regions far distant from the origin (asymptotes, etc.);
2. how to test whether a given point on the curve is a multiple point;
3. the utility of the information derivable from the differential coefficient $\dfrac{dy}{dx}$.

Illustrate by sketching the curves

$$y = \frac{x(x-1)(x-3)}{(x-2)(x-4)}, \quad y^2 = \frac{x^2(x-3)}{x^2-1}.$$

Question (1929 STEP I Q105)

Trace the curve $4(x^2 + 2y^2 - 2ay)^2 = x^2(x^2 + 2y^2)$ and find the radii of curvature of the two branches at the origin.

Question (1940 STEP I Q107)

Find the Cartesian equation of the curve assumed by a uniform string hanging freely under gravity.

If a uniform string is in a vertical plane and is in contact with a smooth horizontal cylinder of any form of cross-section, so that the plane of the string is perpendicular to the generators, show that the difference in the tensions of the string at any two points is proportional to the vertical distance between these points.

If the string lies across a number of such cylinders in a vertical plane perpendicular to their generators, show that the catenaries in which the free portions of the string lie all have the same directrix. Show also that the free ends must be at the same level.

Question (1917 STEP I Q105)

The equation of a rational algebraic curve of the nth degree being written in the form

$$x^n f_0\left(\frac{y}{x}\right) + x^{n-1} f_1\left(\frac{y}{x}\right) + x^{n-2} f_2\left(\frac{y}{x}\right) + \cdots = 0,$$

obtain the asymptotes of the curve; and find the conditions that it may have (i) an inflexional asymptote, (ii) two parallel asymptotes, (iii) a parabolic asymptote. Trace the curve $x(y^2 - 4x^2) - x - 2y + 3 = 0$, and shew that $(\frac{1}{2}, 2)$ is a double point.

Question (1919 STEP I Q106)

Give a systematic account of the rectilinear asymptotes of plane curves, illustrating it by examples. Give a sketch of the curve

$$xy^2 - x^2 - 2y^2 + 3y + 8 = 0.$$

Question (1923 STEP I Q204)

The footway of a suspension bridge is horizontal, and is suspended by vertical rods attached at equal intervals along it. The upper ends of the rods are attached to a light cable. The tensions in all the vertical rods are equal. Show that the points of attachment to the cable lie on a parabola.

Question (1925 STEP I Q204)

$PA_1A_2...A_{2n}Q$ is the chain of a suspension bridge. Each of the vertical bars $A_1B_1, A_2B_2, ...A_{2n}B_{2n}$ bears an equal portion of the weight of the roadway. The distances $B_0B_1, B_1B_2, ...B_{2n}B_{2n+1}$ are all equal. The weights of the chain and bars may be neglected in comparison with the weight of the roadway. By means of a force diagram, or otherwise, shew that the points $P, A_1, A_2, ...A_{2n}, Q$ lie on a parabola whose axis is vertical. If W is the total weight of roadway supported by the bars $A_1B_1, ...A_{2n}B_{2n}$, d the depth of A_nA_{n+1} below PQ, and l the total span of the bridge, shew that the tension in the chain at P or Q is

$$\frac{W}{2}\sqrt{1+\frac{(n+1)^2l^2}{4(2n+1)^2d^2}}.$$

Question (1926 STEP I Q204)

A uniform heavy horizontal beam is supported at its two ends A, B and carries a weight W at C, where $AC = 2CB$. Shew that, if an ordinate y is drawn proportional to the bending moment at P in the rod, where $AP = x$, then the curve obtained consists of portions of two equal parabolas with axes vertical. If the weight of the beam is also W and if $AB = l$, find the latus rectum of the parabolas.

Question (1932 STEP I Q207)

The total mass of a train is 384 tons and the maximum tractive force exerted by the engine at its wheels is 12 tons wt. This force is exerted until the speed of the train from rest reaches 20 feet per second, after which the engine exerts a constant horse power at its wheels. The resistance of the train at various speeds is given by the following table:

Question (1914 STEP II Q210)

The coordinates of points on a curve are given as functions of a parameter θ, prove that in general the condition for an inflexion is $x'y'' - y'x'' = 0$; and that the conditions for a cusp are $x' = 0, y' = 0$, where accents denote differentiation with respect to θ. Shew that the coordinates of a point on the curve $r = a + b\cos\theta$ can be written in the forms $x = \frac{1}{2}b + a\cos\theta + \frac{1}{2}b\cos 2\theta$, $y = a\sin\theta + \frac{1}{2}b\sin 2\theta$; and prove that there are two inflexions if a lies between b and $2b$, while there is a cusp if $a = b$.

Question (1916 STEP II Q209)

State sufficient conditions for $f(x)$ to be a maximum when $x = a$. Show that the angle ϕ between the radius vector and tangent to a curve is a maximum or minimum at any point where the radius of curvature ρ subtends a right angle at the pole; show also that, omitting the case where the curve passes through the pole, ϕ is a maximum if $\frac{d\rho}{ds} - \cot\phi > 0$ at the point.

Question (1921 STEP II Q207)

Show that the function

$$\frac{\sin^2 x}{\sin(x-a)\sin(x-b)}$$

where a, b lie between 0 and π, has an infinity of minima equal to 0 and of maxima equal to $-4\sin a \sin b/\sin^2(a-b)$. Sketch the graph of the function.

Question (1923 STEP II Q208)

Draw the graph from $x = 0$ to $x = \pi$ of

$$y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x.$$

Find the maximum and minimum points on the curve.

Question (1929 STEP II Q209)

Sketch the curve whose equation is

$$y^2 = c^2\frac{(x-a)}{(b-x)} \quad (b > a)$$

and shew that the area enclosed by the curve and its asymptote is $\pi c(b-a)$.

Question (1930 STEP II Q210)

Give a rough sketch of the curve $y^2 = x^5(a-x)(b-x)$, where $0 < a < b$. Shew that if a/b is small, the area of the loop is $\frac{\pi a^4\sqrt{b}}{2^8}\left(20 - \frac{7a}{b}\right)$ approximately.

None

Question (1931 STEP II Q206)

Examine the function
$$\frac{(x+1)^5}{x^5+1}$$
for maxima and minima and sketch the general shape of its graph. Prove that if m does not lie between 0 and 16, the equation
$$(x+1)^5 = m(x^5+1)$$
has no real root other than -1.

Question (1932 STEP II Q206)

Prove that the function
$$\frac{\sin^2 x}{\sin(x-\alpha)},$$
where $0 < \alpha < \pi$, has infinitely many maxima equal to 0 and minima equal to $\sin\alpha$. Sketch the graph of the function.

Question (1933 STEP II Q206)

Shew that there are three points of inflexion on the curve
$$y = \frac{x}{x^2+x+1}.$$
Shew that these three points of inflexion lie on a line.

Question (1934 STEP II Q204)

The function $\cot\theta + k\sec\theta$, $(k > 0)$, has a turning value when $\theta = \alpha$. Find a cubic satisfied by $\sin\alpha$, and shew, by a graph or otherwise, that just one root of this cubic gives real values of α.

Shew that of the two turning values of the function between 0 and π one is a minimum and the other a maximum, and sketch the graph of $\cot\theta + 2\sec\theta$ for the range $-\pi < \theta < \pi$.

Question (1934 STEP II Q208)

Give an account of the method of finding the asymptotes of the curve $P(x, y) = 0$, where P is a polynomial in x and y.

Shew that $x - y = 3$ is an asymptote of
$$(x-y+1)(x-y-2)(x+y) = 8x - 1,$$
find the other asymptotes, and sketch the curve.

Question (1935 STEP II Q204)

Prove that, if k is real and $|k| < 1$, the function $\cot x + k \csc x$ takes all values as x varies through real values. Prove that, if $|k| > 1$, the function takes all values except those included in an interval of length $2\sqrt{k^2 - 1}$. Give rough sketches of the graph of

$$y = \cot x + k \csc x$$

for $-\pi < x < \pi$, in the cases (i) $0 < k < 1$, (ii) $k > 1$.

Question (1935 STEP II Q209)

Find the equation of the straight line which is asymptotic to the curve

$$x^2(x - y) + y^2 = 0.$$

Prove also the following facts and give a sketch of the curve:

1. the origin is a cusp;
2. no part of the curve lies between $x = 0$ and $x = 4$;
3. the curve consists of two infinite branches, one lying in the first quadrant and the other in the second and third quadrants.

Question (1936 STEP II Q210)

Find the asymptotes of the curve

$$(x + y - 1)^3 = x^3 + y^3,$$

and prove that they meet the curve only at infinity. Prove also that there is no point on the curve for values of x between a and 1, where a is the real root of the equation

$$3a^3 + 3a^2 - 3a + 1 = 0.$$

Give a sketch showing the general form of the curve.

Question (1939 STEP II Q207)

Discuss the maxima and minima of the function $\frac{(x-a)(x-b)}{x}$ when $a < b$.

Draw rough graphs illustrating the behaviour of the function for different values of a and b.

Question (1940 STEP II Q206)

Sketch the curves

$$\text{(i) } y = x^2 - x^3; \quad \text{(ii) } y^2 = x^2 - x^3,$$

and find their radii of curvature at the origin.

Question (1940 STEP II Q209)
If $y = x(1-x)/(1+x^2)$,

(i) find the maximum and minimum values of y;

(ii) find the points of inflexion of the curve;

(iii) sketch a graph showing clearly the points determined in (i), (ii), and also the position of the curve relative to the line $y = x$.

Question (1941 STEP II Q208)
A particle moves in a plane so that its position at time t, referred to fixed rectangular cartesian axes, is given by $x = a\sin 2pt$, $y = a\sin pt$. Sketch the path traced out by the particle, and find the radii of curvature at the points where the particle is moving in a direction parallel to one or other of the axes.

Question (1942 STEP II Q203)
Sketch the curve

$$y = x^2/(x^2 + 3x + 2).$$

By means of the line $y + 8 = m(x+1)$, or otherwise, find the number of real roots of the equation

$$m(x+1)^2(x+2) = (3x+4)^2,$$

when m is a real constant which is (i) positive, (ii) negative.

Question (1920 STEP III Q212)
Find the maximum and minimum values of

$$(x+3)^2(x-2)^3,$$

and draw a rough graph of the function. Hence, or otherwise, prove that the equation

$$x^5 - 15x^3 + 10x + 60 = 0$$

has two real negative roots and two imaginary roots.

Question (1913 STEP IIII Q207)
Find the asymptotes of

$$x^2(y+a) + y^2(x+a) + a^2(x+y) = 0,$$

and trace the curve.

Question (1915 STEP IIII Q206)

Prove that the following definitions of the curvature of a curve at a point P lead to the same value.

(i) The rate, in radians per unit of arc, at which the tangent turns.

(ii) The limit to which the reciprocal of the radius of a circle touching the curve at P and passing through an adjacent point Q tends, as Q tends to P.

Examine the nature of the evolute of a given curve in the neighbourhood of the following points on the given curve

(a) a point of inflexion,

(b) a cusp,

(c) a point of maximum or minimum curvature.

Trace the curve $b(ay - x^2)^2 = x^5$, and shew that the evolute has a point of inflexion corresponding to the origin on the curve.

Question (1919 STEP IIII Q202)

Trace carefully the curves

(i) $y = \frac{x(x-1)}{2x-1}$,

(ii) $y^2 = \frac{x(x-1)}{2x-1}$,

and show how inspection of the former would indicate that the latter consists of two distinct branches.

Question (1922 STEP IIII Q204)

Prove that the curve $2x^2 = ay(3x - y)$ has two tangents in the direction of the axis of x and one tangent in the direction of the axis of y, giving the equation in each case. Sketch the curve, and prove that the area of its loop is $81a^2/80$.

Question (1923 STEP IIII Q204)

Discuss the general form of the curve $y = x - a\log(x/b)$, where a and b are positive, and give a rough sketch of the curve. Find the asymptote. Prove that at any point P the chord of curvature, parallel to the asymptote, is proportional to the square of the length PT of the tangent intercepted between P and the asymptote. Give geometrical constructions for the point T and for the centre of curvature at P.

Question (1930 STEP IIII Q205)

A curve C touches the x-axis at the origin. Obtain the expansions

$$x = s - \frac{1}{6}\kappa^2 s^3 + \dots, \quad y = \frac{1}{2}\kappa s^2 + as^3 + \dots,$$

where κ is the curvature $(= 1/\rho)$. Find the coefficient a. Hence shew that for a short arc of length s

$$\text{arc} - \text{chord} = \frac{1}{24}\kappa^2 s^3,$$

if higher powers of s are neglected. Let the tangent at P on the curve cut the x-axis in T, and suppose that C has a point of inflexion at O, but $d\kappa/ds \neq 0$. Shew that

$$\lim_{P \to O} TO/PT = 2.$$

Question (1937 STEP I Q310)

Sketch the curve

$$y(y+1)(y+2) - (x-2)x(x+2) = 0$$

and prove that the point $(0, -1)$ is a point of inflexion.

Question (1938 STEP I Q305)

Find the number of stationary values of the function $y = x^2 + 6\cos x$, distinguishing between maxima and minima, and find the number of points of inflexion.

Question (1940 STEP I Q310)

Trace the curve

$$b^3 y^2 (2 - by) - x^2 = 0,$$

and show that its area is $\dfrac{5\pi}{4b^2}$.

Question (1941 STEP I Q310)

Trace the curve

$$y^2 + 2(x^2 - 2)xy + x^4 = 0,$$

and find the areas of the loops.

Question (1917 STEP II Q308)

Find the limiting value of $(1 - x)^{\log x}$ when $x \to 0$. The equation of a curve is

$$x^2(x + y) + x - y + 1 = 0.$$

Find the asymptotes and trace the curve.

Question (1918 STEP II Q307)

Find the asymptotes of the curve

$$x(x^2 - y^2) + x^2 + y^2 + x + y = 0.$$

Shew that the asymptotes meet the curve again in three points on a straight line, and find the equation of the line.

Question (1925 STEP II Q307)

Find the asymptotes of the curve

$$2x(y-3)^2 = 3y(x-1)^2$$

and trace the curve.

Question (1926 STEP II Q306)

Prove that the radius of curvature at any point of a plane curve is

$$\frac{\{1 + (\frac{dy}{dx})^2\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

A curve is determined by the property that the tangent to the curve at any point P meets a fixed straight line in R, so that the length PR is constant. Prove that the radius of curvature of the curve at P is $\frac{PR \cdot RN}{PN}$, where PN is the perpendicular from P to the fixed straight line.

Question (1926 STEP II Q308)

Prove that the curve $y = e^{-ax} \cos bx$ lies between the curves $y = e^{-ax}$ and $y = -e^{-ax}$, touching each in turn, and has, if $a = b$, points of inflexion at the points of contact.

Question (1926 STEP II Q310)

Make a rough sketch of the curve

$$y^2 = x^2(3-x)(x-2),$$

and shew that its area is $\frac{5\pi}{8}$.

None

Question (1930 STEP II Q308)

Trace the curve $x^4 + ax^2y - ay^3 = 0$, determining the turning points. Using polar coordinates or otherwise, calculate the area of a loop of the curve $\left[\frac{4}{105}a^2\right]$.

Question (1924 STEP III Q311)

Determine the asymptotes of the curve

$$r \cos 3\theta = a$$

and sketch the curve.

Question (1935 STEP III Q308)

Prove that the radius of curvature of a plane curve may be expressed in the form $r\frac{dr}{dp}$. Shew that if for a curve the segment of the normal between any point on the curve and the corresponding centre of curvature subtends a constant angle α ($\neq \frac{\pi}{2}$) at a fixed point, then for a suitable system of coordinates, $re^{-\phi \tan \alpha}$ is constant, where r is the length of the radius vector, and ϕ one of the angles between it and the tangent to the curve. Discuss the case $\alpha = \frac{\pi}{2}$.

Question (1935 STEP III Q310)

Sketch the curve $ay^2 = x(x-a)(x-b)$, where a and b are both positive. Prove that there are two and only two real points of inflexion. If $a = b$, shew that the area of the loop is $\frac{8}{15}a^2$.

Question (1937 STEP III Q305)

A uniform wire hangs in equilibrium under gravity with its ends attached to two fixed supports on the same level at a distance $2a$ apart; the length of the wire is such that the force exerted by it on either support is a minimum. If c is the parameter of the catenary in which the wire hangs and $x = a/c$, shew that

$$\log_{10}(x+1) - \log_{10}(x-1) = 0.87x,$$

approximately. Find an approximation to the root of this equation, and hence to the length of the wire.

Question (1920 STEP IIII Q302)

Shew graphically the change in the value of the function

$$(x-a)(x-b)/(x-c)(x-d),$$

as x changes from $-\infty$ to $+\infty$, where a, b, c, d are real numbers such that

(i) $a > b > c > d$,

(ii) $a > c > b > d$,

(iii) $a > c > d > b$.

Question (1920 STEP IIII Q308)

Define the curvature of a plane curve, and deduce the expression

$$\pm\frac{d^2y/dx^2}{\{1+(dy/dx)^2\}^{3/2}},$$

for the curvature at a point on the curve $y = f(x)$. Prove that the centre of curvature at the point (x, y) on the curve $ay = x^2$ is the point $(-4x^3/a^2, 3y + a/2)$. Find the coordinates of the points where the locus of centres of curvature cuts the original curve, and shew that at these points the curvature of the locus of centres of curvature is $\sqrt{6}/27a$.

Question (1921 STEP IIII Q302)

Draw graphs of the functions

$$\frac{(x-2)(x-4)}{(x-1)(x-3)},\quad \left\{\frac{(x-2)(x-4)}{(x-1)(x-3)}\right\}^{\frac{1}{2}},\quad \frac{x^3}{x^2+1},$$

and shew how to approximate to them for large values of x.

Question (1923 STEP IIII Q306)

Make rough drawings of the curves (i) $y = \dfrac{x^2}{1+x^2}$; (ii) $y = \dfrac{1-x+x^2}{1+x+x^2}$; (iii) $y = \lim_{n\to\infty} \dfrac{x^{2n}\tan\frac{\pi x}{2} + x}{x^{2n}+1}$.

Question (1923 STEP IIII Q308)

Find a formula for the radius of curvature at a point on a curve $\phi(x, y) = 0$. Prove that the equation of the evolute of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is

$$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

Question (1916 STEP I Q406)

Trace roughly the curves $x^2 - y = 2$ and $(y-3)(x+1) + 8 = 0$ between $x = -4$ and $x = 4$. Use your figure to find their points of intersection.

Question (1917 STEP I Q406)

Draw a rough graph of the curve $8y = x(x-3)(x+5)$ between the points $x = \pm 5$, and hence determine for what range of values of a the equation $x^3 + 2x^2 - 15x - 8a = 0$ has three real roots.

Question (1923 STEP I Q408)

Find the equation of the normal at any point on $y^2 = 4ax$. From any point on this parabola, two normal chords of lengths n, n' are drawn to the curve, prove that the radius of curvature at the given point is equal to $nn'/2a$.

Question (1913 STEP II Q405)

Shew how to determine the asymptotes of an algebraic curve, including the cases in which the curve has (i) asymptotes parallel to an axis, (ii) a pair of parallel asymptotes. Find the asymptotes of the curve

$$2x(x+y)^2 - (x+y)^2(x-2y) + x = 0.$$

Question (1916 STEP II Q405)

Prove graphically that the equation $\theta = \cos\theta$ has only one real root, and that it is given approximately by $\cos\theta = \cdot 8$.

Question (1921 STEP II Q408)

Trace the curve $x^4 - x^2y + y^3 = 0$.

Question (1922 STEP II Q409)

Trace the curve $a^3y^2 = x^4(b+x)$, and find the area of the loop.

Question (1924 STEP II Q410)

Find the areas of the curves

1. $a^2(y-x)^2 = (a+x)^3(a-x)$,
2. $(2x^2+3y^2)^3 = a^2xy^4$.

Question (1932 STEP II Q405)

Find the asymptotes of the curve

$$x^2(x+y) = x + 4y,$$

and trace the curve.

Question (1942 STEP II Q404)

The normals to a parabola at points A, B, C are concurrent in P. If P lies on a fixed straight line, prove that the loci of the orthocentre and circumcentre of the triangle ABC are straight lines.

Question (1915 STEP III Q406)

Find the maximum and minimum values of $y = (x+1)^2(x+3)^3(x+2)$ and draw a rough graph of the curve.

Question (1915 STEP III Q407)

Prove that the radius of curvature at any point of a curve is given by

$$\rho = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{x'y'' - y'x''},$$

where accents denote differentiations with regard to a parameter.

Find the curvature at the origin of each of the branches of the curve

$$x^3 - x^2y - xy^2 + y^3 = 4xy - 2y^2,$$

and trace the curve.

Question (1917 STEP III Q410)

Prove that, in the curve $y^2(a+x) = x^2(a-x)$, the area between the curve and its asymptote and the area of the loop are in the ratio $4+\pi : 4-\pi$.

Question (1918 STEP III Q410)

Trace the curve $x^3 + y^3 - 2ax^2 = 0$.

Question (1930 STEP III Q408)

(i) Find the asymptotes and points of inflexion of the curve $y^2(x^2-1) = x^3$. Sketch the curve. (ii) Find the value of $(\cos x)^{1/x^2}$ as x tends to zero.

Question (1938 STEP III Q408)

Find the asymptotes of the curve $xy^2 = 4(x-a)(x-b)$, where $b > a > 0$. Sketch the curve, and find the area bounded by the curve and the lines $x = 0, x + b = 0$.

Question (1939 STEP III Q406)

Trace the curve given by the equation

$$a^3(y+x) - 2a^2x(y+x) + x^5 = 0.$$

Question (1940 STEP III Q405)

If $x > 0$, prove that $(x-1)^2$ is not less than $x(\log x)^2$.

Discuss the general behaviour of the function $(\log x)^{-1} - (x-1)^{-1}$ for positive values of x and with special reference to $x = 1$.

Sketch the graph of the function.

Question (1940 STEP III Q408)

Find the equation of the normal and the centre and radius of curvature of the curve $ay^2 = x^3$ at the point (at^2, at^3).

Shew that the length of the arc of the evolute between the points corresponding to $t = 0$ and $t = 1$ is $\dfrac{13\sqrt{13}-8}{6}a$.

Question (1942 STEP III Q405)

If $y = a^{x^x}$, where a is a positive constant, prove that y has a minimum value and that x has a maximum value. Find the limit of y as $x \to 0$ through positive values, and sketch the graph of y for positive values of x.

Question (1914 STEP IIII Q407)

Find the maximum and minimum values of

$$y = (x-1)^2(x-2)^3(x-3)$$

and draw a rough graph of the curve.

Question (1921 STEP II Q507)

Sketch very roughly the graph of $\sin^2 x$, and show that the equation $x - 2\sin^2 x = 0$ has three real roots and only three, x being measured in radians.

Question (1923 STEP II Q509)

Shew how to find the points of inflexion of the curve $y = f(x)$. Find the maximum point and the inflexions of the curve $y = xe^{-x}$, and trace the curve.

Question (1925 STEP II Q507)

Explain the term 'point of inflexion' of a plane curve, and prove that if $y = f(x)$ has a point of inflexion whose abscissa is x_0, then $f''(x_0) = 0$. The graph of a polynomial of the fourth degree in x touches the x-axis at $(a, 0)$ and has a point of inflexion at $(-a, 0)$. Prove that the graph passes through $(-2a, 0)$ and that it has a second point of inflexion whose abscissa is $a/2$.

Question (1926 STEP II Q509)

Find the asymptotes of the curve $x^2y + xy^2 = x^2 - 4y^2$, and trace it. Find the cubic which has $x + y = 1$ as an asymptote and touches both axes at the origin, the radii of curvature there being 1 and 2 units in length.

Question (1930 STEP II Q506)

Trace the curve

$$(x^2 - y^2)^2 - 4y^2 + y = 0.$$

Question (1922 STEP III Q510)

Find the asymptotes of the curve

$$y^2 = \frac{a^3x}{a^2 - x^2}$$

and find the radius of curvature at the origin. Sketch the curve.

Question (1924 STEP III Q510)

Find the asymptote of the curve

$$x^3 + y^3 = 3axy.$$

Sketch the curve, and by transferring to polar coordinates or otherwise prove that the area of its loop is $\frac{3}{2}a^2$.

Question (1932 STEP III Q507)

Write an account of the theory of rectilinear asymptotes of a plane curve whose equation is given either in rectangular cartesian form or in polar form. The lines whose equations are $x = y, x + y = 0, x = 2y$ are the asymptotes of a cubic curve which touches the axis of x at the origin and which passes through the point $(0, b)$. What is the equation of the curve?

Question (1916 STEP IIII Q508)

Trace the curves (i) $y^2(a - x) = x^3$, (ii) $r = a + b\cos\theta$ $(b > a)$.

Question (1923 STEP IIII Q505)

Shew that the function $\sin x + a\sin 3x$ for values of x from 0 to π has no zeroes except the terminal ones if $-\frac{1}{9} < a < 1$. Shew also that it has two minima with an intermediate maximum if $a < -\frac{1}{9}$, one maximum if $-\frac{1}{9} < a < \frac{1}{9}$, two maxima and an intermediate minimum if $a > \frac{1}{9}$. Indicate roughly the types of the graph of the function in the four cases.

Question (1925 STEP IIII Q507)

Give an account of some method of finding the rectilinear asymptotes of a curve whose x, y equation is given and show how to determine in what manner the curve approaches an asymptote. Consider the cases

(i) $y(x^2 - 1) = x^3 + x^2 + 1$,

(ii) $x(y - x)^2 - 3y(y - x) + 2x = 0$.

Question (1926 STEP IIII Q505)

Prove that the following definitions of the curvature of a curve at a point P lead to the same value.

(i) The rate, in radians per unit of arc, at which the tangent turns.

(ii) The limit to which the reciprocal of the radius of a circle touching the curve at P and passing through an adjacent point Q tends, as Q tends to P.

Examine the nature of the evolute of a given curve in the neighbourhood of the following points on the given curve

(a) a point of inflexion,

(b) a cusp,

(c) a point of maximum or minimum curvature.

Trace the curve $b(ay - x^2) = x^3$, and shew that the evolute has a point of inflexion corresponding to the origin of the curve.

Question (1923 STEP II Q605)

Prove that as the real variable x changes steadily from $-\infty$ to $+\infty$, the function

$$y = \frac{(x-a)^2}{x-b}$$

(where a and b are real and $a < b$) assumes twice over all values except those in an interval of length $4(b-a)$, and locate this range of values precisely.

Question (1924 STEP II Q608)

Show how to find the asymptotes of an algebraic curve without discussing exceptional cases. Find the asymptotes of the curve $x^2y + xy^2 + xy + y^2 + 3x = 0$. Trace the curve.

Question (1927 STEP II Q612)

Find the area of the loop of the curve

$$4y^2 = (x-1)(x-3)^2,$$

and shew that the centroid of this area is at a distance $1\frac{6}{7}$ from the origin.

Question (1927 STEP III Q602)

The normal at P to a parabola whose focus is S cuts the axis in G. Prove that the locus of the middle point of PG is a parabola whose vertex is S.

Question (1922 STEP IIII Q605)

Shew that the function $\sin x + a\sin 3x$ for values of x from 0 to π has no zeros except the terminal ones if $-\frac{1}{3} < a < 1$. Shew also that it has two minima with an intermediate maximum if $a < -\frac{1}{3}$, one maximum if $-\frac{1}{3} < a < \frac{1}{3}$, two maxima and an intermediate minimum if $a > \frac{1}{3}$. Indicate roughly the types of the graph of the function in the four cases.

Question (1918 STEP II Q712)

Trace the curve $y = e^{1/x}$. Find the inflexions and the asymptotes.