Question (1966 STEP II Q6)

Show that $y = \sin x \tan x - 2 \log \sec x$ increases steadily as x increases from 0 to $\frac{1}{2}\pi$. Show also that y has no inflexion in this range. Sketch the curve $y(x)$ in

$$0 \leq x < \frac{1}{2}\pi.$$

Question (1968 STEP II Q13)

By means of the calculus or otherwise, prove that if $p > q > 0$ and $x > 0$, then

$$q(x^p - 1) > p(x^q - 1).$$

Hence or otherwise prove that, under the same conditions,

$$\frac{1}{p}\left(\frac{x^p}{(p+1)^p} - 1\right) > \frac{1}{q}\left(\frac{x^q}{(q+1)^q} - 1\right)$$

for every positive integer n.

Question (1966 STEP IIII Q4)

It is given that

$$f_n(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots + \left(\frac{1}{n}\right)\sin nx$$

For each integer $n = 1, 2, 3, \ldots$ If x_0 is any minimum of $f_n(x)$ in the range $0 < x < \pi$, prove that $\sin x_0 < 0$, and hence that $\sin x_0$ and $\sin(n + \frac{1}{2})x_0 = \sin \frac{1}{2}x_0$. Deduce, by using mathematical induction on n, that $f_n(x)$ can never take negative or zero values in the range $0 < x < \pi$, for any $n \geq 1$.

Question (1973 STEP IIII Q8)

Prove that, if $0 < x < 1$,

$$\pi < \frac{\sin \pi x}{x(1-x)} \leq 4.$$

Question (1979 STEP IIII Q7)

Show that $e^{-t^2/2} \geq \cos t$ for $0 \leq t \leq \frac{1}{4}\pi$.

Question (1979 STEP IIII Q9)

The function $\log^+(x)$ is defined by

$$\log^+(x) = \begin{cases} \log_e(x) & (x \geq 1) \\ 0 & (x < 1) \end{cases}$$

Positive numbers $\lambda_1 > \lambda_2 > ... > \lambda_n$ and $\mu_1 > \mu_2 > ... > \mu_n$ satisfy

$$\lambda_1\lambda_2...\lambda_j \geq \mu_1\mu_2...\mu_j \quad \text{for} \quad 1 \leq j \leq n.$$

Show that

$$g(x) = \sum_{j=1}^{n} \log^+(\lambda_j x) \geq h(x) = \sum_{j=1}^{n} \log^+(\mu_j x),$$

for all x. By considering

$$\int_0^\infty \frac{g(x)}{x^{s+1}}dx \quad \text{and} \quad \int_0^\infty \frac{h(x)}{x^{s+1}}dx,$$

show that

$$\lambda_1^s + ... + \lambda_n^s \geq \mu_1^s + ... + \mu_n^s, \quad \text{for} \quad s > 0.$$

Question (1963 STEP IIII Q205)

Prove that the positive number a has the property that there exists at least one positive number that is equal to its own logarithm to the base a if and only if $a \leq e^{1/e}$.

Question (1960 STEP IIII Q310)

Let

$$f_m(x) = \frac{x}{2}\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots + (-1)^{m+1}\frac{\sin mx}{m}\right].$$

By considering $df_m(x)/dx$, or otherwise, show that

$$(-1)^m f_m(x) > 0$$

for $0 < x < \pi/(2m+1)$. Show also that

$$(-1)^m f_m\left(\frac{\pi}{m+\frac{1}{3}}\right) < 0.$$

Question (1961 STEP II Q102)

Find the ranges of values of x for which the function $(\log x)/x$ (i) increases, (ii) decreases, as x increases. Hence determine the largest possible value of the positive constant k such that the inequalities $0 < x < y < k$ imply that $x^y < y^x$.

Question (1952 STEP IIII Q105)

The polynomial $P(x)$ is defined, for a given positive integer n, by

$$P(x) = \frac{d^n y}{dx^n},$$

where $y = (x^2 - 1)^n$. Find the values of $P(0)$, $P(1)$, $P(-1)$. Prove that

$$(x^2 - 1)P''(x) + 2xP'(x) - n(n+1)P(x) = 0.$$

Question (1955 STEP IIII Q307)

α is a real number and

$$\frac{\alpha x - x^3}{1 + x^2}$$

is increasing for all real x. Show that

$$\alpha \geq \frac{9}{8}.$$

Question (1952 STEP II Q408)

Find for what ranges of x the function $\dfrac{\log x}{x}$ increases as x increases, and decreases as x increases. Hence show that if n is a given positive number and x is a positive real variable, the equation $x = n^x$ has two roots, one root, or no root according to the value of n, and state the critical values of n concerned.

Question (1957 STEP II Q407)

Define $\log_e x$ for $x > 0$. Prove that for $x > 1$:

$$x^2 - x > x \log_e x > x - 1 \quad \text{and} \quad x^2 - 1 > 2x \log_e x > 4(x-1) - 2\log_e x.$$

Question (1945 STEP IIII Q104)

Prove that, if a is real, the equation

$$e^x = x + a$$

has two real roots if a is greater than 1 and no real roots if a is less than 1. Prove that the equation has no root of the form iv, where v is real and not zero, and that, if $u + iv$ is a complex root, u is positive.

Question (1944 STEP II Q104) (i) Prove that, for positive values of x,

$$\log(1+x) < \frac{x(2+x)}{2(1+x)}.$$

(ii) Find whether $e^{-x^2}\sec^2 x$ has a maximum or a minimum value for $x=0$.

Question (1945 STEP II Q103)
Prove by differentiation, or otherwise, that

$$xy \le e^{x-1} + y\log y$$

for all real x and all positive y. When does the sign of equality hold?

Question (1914 STEP I Q112)
Shew that

$$f(x) = \frac{1-x}{\sqrt{x}} + \log x$$

has a differential coefficient which is negative for all values of x between 0 and 1. Hence shew that, if $0 < x < 1$,

$$\sqrt{x}\log\left(\frac{1}{x}\right) < 1-x.$$

Question (1921 STEP I Q110)
The area of a triangle ABC is calculated from the measured values a, b of the sides BC, CA and the measured value $90°$ of the angle C. It is found that the calculated area is too large by a small error z, and that the true lengths of the sides are $a-x, b-y$, where x and y are small. Shew that the error in the angle C is approximately $\frac{180}{\pi}\left(\frac{2z-ay-bx}{\frac{1}{2}ab}\right)$ degrees.

Question (1923 STEP I Q110)
Differentiate $\sin^{-1}\{2x\sqrt{(1-x^2)}\}$, $a^{x\log a}$. If x is large, show that the differential coefficient of $(1+\frac{1}{x})^x$ is approximately $e/2x^2$.

Question (1939 STEP I Q105)

Prove that

$$(3\cos\phi - \sec\phi)^2 \geq 12$$

for all real values of ϕ.

Use this result to show that

$$\frac{\sin\theta - \cos\theta}{3\cos\theta + 3\cos\phi - \sec\phi}$$

lies between $1 - \sqrt{(5/3)}$ and $1 + \sqrt{(5/3)}$ for all real values of θ and ϕ.

Question (1940 STEP I Q107)

Show that the equation

$$\frac{d^n}{dx^n}\left(\frac{1}{x^2+1}\right) = 0$$

has just n roots (all real), and determine them.

Question (1916 STEP I Q112)

Prove that, if f is a homogeneous polynomial in x and y, of degree n, then

(1) $xf_x + yf_y = nf$,

(2) $xf_{xx} + yf_{xy} = (n-1)f_x$,

(3) $\begin{vmatrix} f_{xx} & f_{xy} & f_x \\ f_{xy} & f_{yy} & f_y \\ f_x & f_y & 0 \end{vmatrix} + \frac{nf}{n-1}(f_{xx}f_{yy} - f_{xy}^2) = 0$,

suffixes denoting differentiations with respect to x and y.

Question (1917 STEP I Q111)

Shew, by use of the methods of the differential calculus, or otherwise, that

$$\frac{1}{2} < \frac{e^x}{e^x - 1} - \frac{1}{x} < 1$$

for all positive values of x.

Question (1927 STEP I Q103)

Prove that if $x + y + z = a$, where a is a given positive number, the function

$$u = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy$$

has the minimum value $-\frac{1}{2}a^2$ and no maximum. Prove also that if x, y, z are further restricted to be not negative, the maximum value of u is a^2.

Question (1919 STEP I Q113)

Find the equation of the tangent from the origin to the curve

$$y = e^{\lambda x} \quad (\lambda > 0).$$

Deduce, or prove otherwise, that the equation $x = e^{\lambda x}$ has 0, 1, or 2 real roots according as $e\lambda > 1, e\lambda = 1, e\lambda < 1$.

Question (1927 STEP I Q104)

Assuming that if $f'(x)$ is positive $f(x)$ increases with x, and that if $f'(x)$ is negative $f(x)$ decreases as x increases, prove that for $p > q$ (p, q positive or negative)

$$\frac{x^p - 1}{p} > \frac{x^q - 1}{q},$$

when $x > 1$; and obtain the corresponding inequality for $0 < x < 1$. Prove further, without assuming the logarithmic series, that for x positive but otherwise unrestricted,

$$\log(1+x) \gtrless x - \frac{x^2}{2} + \cdots + \frac{(-1)^n x^n}{n} \quad \left(\begin{smallmatrix} > \text{ when } n \text{ is even} \\ < \text{ when } n \text{ is odd} \end{smallmatrix}\right),$$

and that for $-1 < x < 0$,

$$x - \frac{x^2}{2} + \cdots + \frac{(-1)^n x^n}{n} > \log(1+x) > x - \frac{x^2}{2} + \cdots - \frac{(-1)^n x^n}{n(1+x)}.$$

Deduce the logarithmic series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

for $-1 < x \leq 1$.

Question (1933 STEP I Q104)

Explain how to determine the maximum and minimum values of a function of a single real variable by means of its differential coefficient. Illustrate by considering the function $(x-2)^5(x-3)^{10}$. Shew that $\tan 3x \cot 2x$ takes all values except those between $\frac{2}{3}$ and $\frac{9}{4}$.

Question (1918 STEP I Q103)

Prove the formula

$$f(x+h) - f(x) = hf'(x+\theta h),$$

where $0 < \theta < 1$. Deduce that a function of x whose differential coefficient is positive increases steadily with x. Prove that the functions

$$\log(1+x) - \frac{x}{1+x}, \quad (2+x)\log(1+x) - 2x$$

are positive for all positive values of x; and shew how this sequence of functions may be continued further.

Question (1915 STEP II Q201)

Prove that, if $y = (ax+b)/(cx+d)$, there are two values of x which are equal to the corresponding values of y, and that these are real and distinct, coincident, or imaginary according as $(a+d)^2 >=$ or $< 4(ad-bc)$. Shew that, if these values are h, k, the relation between y and x can be put in the form $\frac{y-h}{y-k} = \lambda\frac{x-h}{x-k}$.

Question (1924 STEP II Q206)

Prove that the least value of $a\cos\theta + b\sin\theta$ is the negative square root of a^2+b^2. Prove also that the least value of

$$x^2 + 2x(a\cos\theta + b\sin\theta) + c\cos 2\theta + d\sin 2\theta$$

is

$$-\frac{1}{2}(a^2+b^2) - \{c^2+d^2+\frac{1}{4}(a^2+b^2)^2 + c(b^2-a^2) - 2abd\}^{1/2}.$$

Question (1925 STEP II Q207)

Draw the graph of the function $a\csc x + b\sec x$ for values of x between 0 and 2π, taking a, b to be positive. Determine the number of real roots between 0 and 2π of the equation $a\csc x + b\sec x = 1$, distinguishing the cases

$$a^{2/3} + b^{2/3} >, =, < 1.$$

Question (1927 STEP II Q207)

Find the equation determining the values of x for which $\dfrac{\sin mx}{\sin x}$ is stationary. Hence, or otherwise, shew that, if m is an integer, $\dfrac{\sin^2 mx}{\sin^2 x}$ never exceeds m^2.

Question (1936 STEP II Q207)

Prove that, if x is positive,

$$\frac{2x}{2+x} < \log(1+x) < x.$$

Prove also that, if a and h are positive,

$$\log(a+\theta h) - \log a - \theta\{\log(a+h) - \log a\},$$

considered as a function of θ, has a maximum for a value of θ between 0 and $\frac{1}{2}$.

Question (1937 STEP II Q202)

Prove that for real values of x the rational function

$$\frac{5x^2 - 18x - 35}{8(x^2-1)}$$

takes all real values except those between 1 and 4. Draw a rough graph of the function.

Question (1916 STEP IIII Q207)

Shew from the differential coefficients that the functions

$$x - \log(1+x), \quad \frac{2x}{2+x} - \log(1+x)$$

are respectively positive and negative when x is positive. Shew also that, when a and h are positive, $\log(a+\theta h) - \log a - \theta\{\log(a+h) - \log a\}$ considered as a function of θ, has a maximum for a value of θ between 0 and $\frac{1}{2}$.

Question (1919 STEP IIII Q204)

Determine for what ranges of x the function $(\log x)/x$ (i) increases and (ii) decreases as x increases. Hence, or otherwise, prove the following theorems wherein n is a given positive number and only positive values of x are considered:

(I) The equation $n^x = x$ has two roots, one root, or none according to the value of n.

(II) The inequality $n^x > x^n$ is a consequence of either $x > n$ or $x < n$ according to the value of n.

State the critical values of n.

Question (1919 STEP I Q308)
Shew that

$$f(x+h) - f(x) = hf'(x+\theta h)$$

for some value of θ lying between 0 and 1, provided $f(x)$ and its differential coefficient $f'(x)$ satisfy certain conditions which are to be stated. Deduce that a function of x whose differential coefficient is positive, increases steadily as x increases. Hence shew that if $0 < x < \frac{\pi}{2}$

$$1 > \cos x > 1 - \frac{x^2}{2} \quad \text{and} \quad x > \sin x > x - \frac{x^3}{6}.$$

Question (1939 STEP I Q305)
Find all maxima and all minima of the two functions

$$y = e^{-\sqrt{3}x} \sin^3 x$$

and

$$y = \int_x^\infty \frac{\sin \xi}{\xi(1+\xi^2)} d\xi.$$

Question (1940 STEP I Q301)
Prove that, if $x > 0, 0 < p < 1$, then

$$(1+x)^p < 1 + px.$$

Hence show that, if $a > 0, b > 0$ and if n is a positive integer, then

$$(a^n + b)^{1/n} < a + \frac{b}{na^{n-1}}.$$

Show further by using the identity

$$(a^n + b) - a^n = b$$

$$(a^n + b)^{\frac{n-1}{n}} + a(a^n + b)^{\frac{n-2}{n}} + \cdots + a^{n-1}$$

or otherwise, that

$$a + \frac{b}{na^{n-1} + \frac{1}{2}(n-1)\frac{b}{a}} < (a^n + b)^{1/n}.$$

Question (1914 STEP III Q314)
Examine whether the function

$$\frac{\sin^3 x}{x^2 \cos x}$$

is a maximum or minimum when $x = 0$.

Question (1933 STEP III Q308)

Prove that
$$\frac{1+2x-x^2+2\sqrt{x-x^3}}{1+x^2}$$
is a maximum or minimum when $x=-1\pm\sqrt{2}$.

Question (1921 STEP II Q405)

Define the differential coefficient of a function of x. If $f(x)$ is positive shew that $f(x)$ is increasing. Prove that $x\log x > x-1$ if x is positive. Differentiate $x^x, \tan^{-1}\left(\frac{x\sin\alpha}{1-x\cos\alpha}\right)$.

Question (1922 STEP II Q406)

Prove that, if y is an implicit function of x satisfying the equation $f(x,y)=0$, then
$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}.$$
If A,B,C are the angles of a triangle and $\sin^2 A+\sin^2 B+\sin^2 C=k$, prove that,
$$\frac{\partial A}{\partial B}=\frac{\tan C-\tan B}{\tan A-\tan C}.$$

Question (1923 STEP II Q406)

Prove that $x=\pi/3$ will make $\cos^{-1}(a\sin x)+2\cos^{-1}(a\cos\frac{x}{2})$ a minimum if $0<a<1$ but a maximum if $a>1$.

Question (1924 STEP II Q405)

If $f'(x)$ is positive shew that $f(x)$ is increasing. Prove that $2x+x\cos x-3\sin x>0$ if $0<x<\frac{\pi}{2}$.

Question (1931 STEP II Q409)

Prove that a continuous function attains its upper bound in an interval. Discuss the continuity of the function
$$\log(1+x)\sin\frac{1}{x}$$
in the interval $(0,1)$.

Question (1925 STEP III Q409)

If α and β are given acute angles, and $\alpha > \beta$, prove that the maximum and minimum values of

$$\frac{1 + 2x\cos\alpha + x^2}{1 + 2x\cos\beta + x^2}$$

are

$$\frac{1-\cos\alpha}{1-\cos\beta} \quad \text{and} \quad \frac{1+\cos\alpha}{1+\cos\beta} \text{ respectively.}$$

Question (1938 STEP III Q407)

If $x > 1$, prove that

$$x^3 + 3x + 2 + 6x\log x > 6x^2,$$
$$x^4 + 8x + 12x^2\log x > 8x^3 + 1.$$

Question (1914 STEP I Q507)

Define a differential coefficient, and shew that if $\frac{dy}{dx}$ is positive for any value of x, the value of y is increasing as x increases through that value. Prove that

$$\tan x > x + \frac{x^3}{3}. \quad \left(0 < x < \frac{\pi}{2}\right)$$

Question (1918 STEP II Q506)

Prove the method of determining and discriminating between maximum and minimum values of a function of a single variable by means of its differential coefficients. If O be the centre of an ellipse whose semi-axes are a and b, ON the perpendicular to the tangent at P, shew that the maximum area of the triangle OPN is $\frac{1}{4}(a^2 - b^2)$.

Question (1914 STEP III Q505)

Find the maximum and minimum values of y, where $y^2 = x^2(x-1)^3$.

Question (1921 STEP III Q510)

Prove that the nth differential coefficient of $e^{ax}\sin bx$ is

$$(a^2+b^2)^{\frac{n}{2}} e^{ax} \sin\left(bx + n\tan^{-1}\frac{b}{a}\right).$$

Prove that the curve

$$y = e^{-x^2}$$

has points of inflexion where $x = \pm\frac{1}{\sqrt{2}}$. Sketch the curve roughly.

Question (1933 STEP III Q506)

Explain the application of the Calculus to the discussion of inequalities, giving simple illustrations. Hence or otherwise prove that if $0 < \theta < 1$,

$$\theta + \frac{\theta^3}{3} < \tan\theta < \theta + \frac{2\theta^3}{3}.$$

Question (1925 STEP IIII Q506)

Show that the function $\sin x + a\sin 3x$ for values of x between 0 and π has two minima with an intermediate maximum if $a < -\frac{1}{9}$; one maximum if $-\frac{1}{9} < a < \frac{1}{9}$; and two maxima with an intermediate minimum if $a > \frac{1}{9}$.

Question (1930 STEP IIII Q505)

Prove that the circle of curvature at a point (x, y) will have contact of the third order with the curve if

$$(1 + y_1^2)y_3 = 3y_1y_2^2$$

where

$$y_1 = \frac{dy}{dx}, \quad y_2 = \frac{d^2y}{dx^2}, \ldots.$$

Find the points on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which have this property.

Question (1924 STEP II Q607)

Show that the function $\sin x + a\sin 3x$ for values of x between 0 and π has two minima with a maximum between, if $a < -\frac{1}{3}$; one maximum, if $-\frac{1}{3} < a < \frac{1}{3}$; two maxima with a minimum between, if $a > \frac{1}{3}$.

Question (1927 STEP II Q609)

Find the maximum and minimum values of the expression $\dfrac{2x^2 - 7x + 3}{x - 5}$. Shew that the least value of the sum of the squares of the perpendiculars from a point within a triangle ABC on the three sides is $\frac{1}{2}(a^2 + b^2 + c^2) - \dfrac{a^4 + b^4 + c^4}{2(a^2 + b^2 + c^2)}$.

Question (1914 STEP III Q610)

If $y = a + x\sin y$, where a is a constant, prove that, when $x = 0$,

$$\frac{dy}{dx} = \sin a, \text{ and } \frac{d^2y}{dx^2} = \sin 2a.$$

Hence by Maclaurin's Theorem expand y in powers of x as far as x^2.

Question (1918 STEP II Q703)

Prove that under certain stated conditions the equation $f(x, y) = 0$ determines y as a unique continuous function of x in a certain x-neighbourhood. Prove that the equation $x^3 + y^3 = 3axy$ has a continuous solution in y as a function of x for real values of x whose modulus exceeds a certain value, and that

$$x + y = -a + a^3/x^2 + O(1/x^3),$$

as $x \to +\infty$ or as $x \to -\infty$. What is the geometrical significance of the last result?