Question (1979 STEP II Q11)

A heavy plane plate is dropped on to two identical parallel horizontal rough rollers whose axes are a distance a apart in the same horizontal plane. The rollers are rotating extremely rapidly and the coefficient of sliding friction μ is constant. Discuss the motion of the plate according to the various senses of rotation of the rollers.

Question (1980 STEP II Q12)

A weightless rod carries a particle of mass m at its upper end. It is balanced in unstable equilibrium on a rough horizontal table, and begins to fall sideways. Using conservation of energy, find the angular velocity (squared) and the angular acceleration as functions of the angle θ through which it has fallen, assuming the lower end does not move. Use these to show that the vertical component of force, where the rod touches the table, is

$$N = mg(3\cos^2\theta - 2\cos\theta),$$

and find the horizontal component. Let the coefficient of friction between the rod and the table be μ. Show that the rod's lower end either leaves the surface of the table when $\cos\theta = \frac{1}{3}$, or slips when $\tan\theta = \mu$. What determines which happens?

Question (1975 STEP III Q14)

A particle can slide smoothly in a uniform straight tube. The tube and the particle have equal masses. The tube can rotate freely in a horizontal plane about a fixed end. It is given an initial angular velocity, and the particle is displaced slightly along the tube from its fixed end. Show that the particle must eventually leave the other end, and does so at an angle of approximately $\tan^{-1}\frac{1}{2}$ to the axis of the tube.

Question (1980 STEP III Q11)

A uniform plank is held at rest with one end on a smooth horizontal floor and with the other end against a smooth vertical wall. The plank makes an angle of $60°$ with the vertical wall. If the plank is released from rest, show that the top end of the plank loses contact with the wall after it has slipped down the vertical wall through a distance equal to $\frac{1}{6}$ of the length of the plank.

Question (1966 STEP IIII Q11)

A particle A of mass m and a particle B of mass $2m$ are connected by a light string of length a and slide on a smooth horizontal table. Initially both are at rest with the string taut, when another particle of mass m moving with velocity U perpendicular to AB embeds itself in A. Show that A comes to rest again after a time $\frac{2ma}{U}$. What is then the velocity of B?

Question (1967 STEP IIII Q10)

A particle moves under a central attractive force $f(r)$ per unit mass when its distance from the centre of force is r. Find the form of $f(r)$ if the particle describes the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the centre of force is at the origin.

Question (1975 STEP IIII Q12)

A massless hoop, of radius a, stands vertically on a rough plane. A weight is attached to the rim of the hoop so that the radius to the weight makes an angle θ_0 $(0 \leq \theta_0 < \pi)$ to the upward vertical. In the subsequent motion the hoop remains vertical and rolling occurs without slipping until the vertical reaction at the point of contact with the plane is zero. Show that this occurs when $\theta = \theta_1$ where $\frac{1}{2}\pi \leq \theta_1 < \pi$. At the moment when the vertical reaction is zero, the plane is removed. Show that the velocity of the weight when it reaches the former level of the plane is

$$2\sqrt{2ag} \cos\left(\frac{\theta_1}{2}\right).$$

Question (1975 STEP IIII Q15)

A long thin pencil is held vertically with one end resting on a rough horizontal plane whose coefficient of static friction is μ. The pencil is released and starts to topple forward making an angle $\theta(t)$ to the vertical. Show that there is a critical value of μ, say μ_1, such that

  (i) if $\mu < \mu_1$ the pencil base slips backwards before $\cos\theta = 9/11$;

  (ii) if $\mu > \mu_1$ the pencil base slips forwards at some value of θ lying in the range $\frac{2}{3} > \cos\theta > \frac{1}{3}$.

Find the value of μ_1.

Question (1980 STEP IIII Q13)

A ring of weight mg is free to move on a fixed smooth horizontal rod. A light inextensible string of length $2l$ is attached to the ring at one end. Its other end is attached to a particle of weight mg. The system is held with the particle just below the rod and with the string just taut and lying along the underside of the rod. The system is released from rest in this position. Express the velocities of the ring and the particle in terms of the angle θ made by the string with the horizontal during motion of the system in which the string remains taut. For such motion, show that

$$l\dot{\theta}^2 = \frac{2g\sin\theta}{1+\cos^2\theta}$$

and evaluate the tension in the string as a function of θ. Hence show that for $0 < \theta < \pi$ the string never becomes slack. Show also that the particle follows an elliptic path.

Question (1980 STEP IIII Q15)

Two planets circle around their common centre of gravity C under the influence of Newtonian gravity; the effect of their parent sun can be neglected. Obtain the total energy of their mutual motion, and the total angular momentum about C, as functions of the planets' masses and separation. Miners from the initially less massive planet take ore from the other planet back to their home planet, at a slow, fairly steady, rate. Assuming the two orbits always remain circular, and that the planets are small compared with their separation, show that as the relative masses of the planets change so will their separation, and it will reach a minimum when the two masses are equal. As the planets start to separate at later times, an ecologist suggests that the journey could be kept short, and hence fuel saved, if equivalent masses of unwanted material were shipped back at the same rate, thus keeping the separation at its minimum. Do you agree that this would save fuel in the long run?

Question (1961 STEP IIII Q108)

A uniform plane lamina has a polygonal boundary and rests on a smooth horizontal table. Forces act at the mid-points of the sides, each directly along the inward normal and represented in magnitude by the length of the side on which it acts. Show that the lamina is in equilibrium. The lamina is now turned through an angle α (less than π) about its centroid, the forces retaining their magnitudes, points of application, and directions in space. If the lamina is now released from rest, show that it will turn about its centroid through an angle $2\pi - 2\alpha$ before coming again to rest, and will return to its initial position at equal intervals of time.

Question (1963 STEP IIII Q110)

A uniform pole of length $2a$, standing vertically on rough ground, is slightly disturbed and begins to fall over. If it has not slipped by the time it makes an angle θ with the vertical, show that

$$\frac{d\theta}{dt} = \left(\frac{3g}{a}\right)^{1/2} \sin\frac{\theta}{2}.$$

Find the horizontal and vertical components of the force exerted by the ground on the pole, as a function of $\cos\theta$, and prove that the pole will certainly have slipped before it can reach a certain angle α, however great the coefficient of friction may be.

Question (1959 STEP II Q309)

A bead moves on a rough wire which is in the shape of the cycloid whose intrinsic equation is

$$s = 4a\sin\psi.$$

The wire is in a vertical plane and its cusps point upwards, s is measured from the lowest point, and ψ is the angle between tangent and horizontal. Show that if the particle is released from rest at one of the cusps it just comes to rest again at the bottom of the wire if the coefficient of friction μ satisfies the equation

$$\mu^2 e^{4\pi} = 1.$$

Question (1964 STEP II Q307)

Two particles A and B, of equal mass, are joined by a light inextensible string. A moves on a rough horizontal table (coefficient of friction μ) and the string passes through a small smooth hole O, so that B hangs below the table. Show that, if (r, θ) are polar coordinates of the position of A relative to O,

$$\frac{d}{dt}(r^2\dot\theta) = -\mu g r^2\dot\theta/v,$$

$$2\ddot r - r\dot\theta^2 = -g(1 + \mu^2/v),$$

where $v^2 = \dot r^2 + r^2\dot\theta^2$ and the dot denotes differentiation with respect to the time t. Initially $r = R$ and the velocity of A is at right angles to the string and of magnitude $\sqrt{(gR)}$. If μ is small an approximate solution to the equations of motion is

$$r = R + \mu\rho(t),$$

$$r\theta = \sqrt{(gR)} + \mu h(t),$$

where the functions $\rho(t)$ and $h(t)$ are independent of μ. Show that

$$h = -gt - \sqrt{(g/R)}\rho$$

and

$$\ddot\rho + \frac{3g}{2R^2}\rho + g\sqrt{\frac{g}{R}}t = 0.$$

Question (1959 STEP III Q110)

A uniform thin straight rod AB, of mass M and length $2l$, is initially at rest on a smooth horizontal table. If the end A is constrained to move from rest with constant acceleration f in a horizontal straight line at right angles to the rod, find the components of the force being exerted on the rod at A at the instant when the rod has turned an angle θ from its initial direction. Discuss whether the rod will make complete revolutions.

Question (1960 STEP III Q107)

Two equal light rods AB, BC are freely jointed at B and lie on a smooth table. A heavy weight is attached at A, and the point C is fixed. The rod BC is constrained to pass through a fixed point. Initially, A is at rest and ABC are in a straight line, and roughly describe the motion of A.

Question (1960 STEP III Q108)

Three equal heavy particles XYZ lie in a straight line on a smooth table. XY and YZ are joined by similar light springs, each of natural length L. Initially, the particles are still, and the distance XY is L, and the distance YZ is $L - y_0$ ($y_0 < L$). Describe the subsequent motion—in particular, show that it is periodic.

Question (1958 STEP III Q206)

A smooth wire AB of length a is originally in a vertical line, B being above A. A stop is attached to the wire very near the end B and a heavy bead is threaded on to the wire just above the stop (so that the bead cannot move nearer to A, but is free to leave the wire after moving a negligibly small distance away from the stop). The wire is then suddenly constrained to rotate with uniform angular velocity in a vertical plane about the end A, which remains fixed. Find where the bead leaves the wire, and at what distance from A it meets the horizontal plane through A.

Question (1960 STEP III Q204)

A particle P of mass m moves in a hyperbolic orbit under the influence of a radial repulsion k/r^2 from a fixed focus O, where $r = OP$. The particle starts at a great distance from O with a speed v along a line to which the perpendicular from O has length b. If u is the speed of the particle when it is closest to O, show from the equations of energy and angular momentum respectively that

$$\left(\frac{u}{v}\right)^2 = 1 - \frac{2k\sin\alpha}{mv^2 b(1+\cos\alpha)}, \quad \frac{u}{v} = \frac{\sin\alpha}{1+\cos\alpha},$$

where α is the acute angle between the initial direction of motion of the particle and the axis of symmetry of the orbit. Deduce that when the particle has receded a great distance from O its direction of motion has turned through the angle

$$2\tan^{-1}\left(\frac{k}{mv^2 b}\right).$$

Question (1960 STEP III Q206)

A light spring $ABCD$, of natural length $3a$ and modulus λ, lies on a smooth horizontal table to the surface of which its ends A, D are fixed at points distant $3a$ apart. Particles of mass m are rigidly attached to the points of trisection B, C of the unstretched spring, and subsequently the particle at B is moved a distance $\frac{1}{2}a$ towards that at C, which is held stationary. The system is then released from rest. Determine the subsequent motion.

Question (1961 STEP III Q205)

A particle P of unit mass moves on a smooth horizontal plane on which Ox, Oy are fixed rectangular cartesian axes. P is attracted towards O with force $n^2 r$, where r is the distance OP. The particle is projected from the point C $(c, 0)$ with velocity nb, in the direction which makes an angle α with Ox. Show that P moves on the ellipse $b^2(x\sin\alpha - y\cos\alpha)^2 + c^2 y^2 = b^2 c^2 \sin^2\alpha$. Using $t = \tan\alpha$ as parameter, or otherwise, show that all points of the plane which can be reached by projection from C with speed nb lie within or on the ellipse $b^2 x^2 + (b^2 + c^2)y^2 = b^2(b^2 + c^2)$.

Question (1961 STEP III Q207)

Two unequal masses, m_1 and m_2, are fixed to the ends of a light elastic spring of length k. The spring is laid on a smooth horizontal table and compressed through a distance l. Both ends are then released simultaneously. Investigate mathematically the subsequent motion of the system.

Question (1962 STEP III Q208)

A uniform rod AB of length $2a$ and mass m stands balanced vertically on a smooth horizontal table, A being the point of contact. A horizontal impulse I is applied at A. Show that if $I > \frac{1}{2}m\sqrt{(ag)}$ the rod leaves the table immediately, and in this case find that value of I for which the rod is horizontal at its first impact with the table.

Question (1958 STEP III Q405)

A narrow straight tube of length $2a$ has one end fixed and is made to rotate in a plane with constant angular velocity ω. A small bead is instantaneously at rest at $t = 0$ at the mid-point of the tube, and the coefficient of friction in the tube is $\frac{1}{3}$. If gravity can be neglected, show that the particle will reach the other end of the tube after time $(2/\omega)\log x$, where x is the larger positive root of the equation $4x^2 - 10x^4 + 1 = 0$.

Question (1959 STEP III Q407)

A smooth rigid wire in the form of a parabola is held fixed in a vertical plane with its vertex downwards. A bead moves under gravity on the wire. Prove that at the square of the normal reaction of the bead on the wire is inversely proportional to the cube of the height of the point above the directrix of the parabola.

Question (1955 STEP IIII Q110)

Two particles, of masses m and $3m$, are joined by a light inextensible string of length $4a$. The system can move freely in a smooth horizontal plane. Initially the string is straight, the heavier particle is stationary, and the lighter particle moves with velocity v at right angles to the string. Describe the subsequent motion, and sketch the paths of the particles. Calculate the maximum speed of the heavier particle in the subsequent motion and the tension in the string.

Question (1956 STEP IIII Q109)

Two masses m_1, m_2 are connected by a light elastic string of modulus λ and natural length l and lie at rest at A, B, respectively, on a smooth horizontal table, where $AB = l$. If an impulse J is applied to m_1 in the direction BA, find the extension of the string when the relative velocity of the two masses first vanishes, and after what time this occurs.

Question (1955 STEP II Q208)

A particle A of mass m_1 is hung from a fixed point O by a string of length l and a particle B of mass m_2 is hung from A by a string of equal length. The strings have negligible weight and are inextensible. The particles are constrained to move in a fixed vertical plane through O, and the inclinations of the upper and lower strings to the vertical, θ_1 and θ_2 respectively, are so small that their squares and products may be neglected. Show that the equations of motion may be written

$$(m_1 + m_2)l\ddot{\theta}_1 + m_2 l\ddot{\theta}_2 + (m_1 + m_2)g\theta_1 = 0,$$
$$l\ddot{\theta}_1 + l\ddot{\theta}_2 + g\theta_2 = 0.$$

Deduce that $\theta_1 + \alpha\theta_2$ and $\theta_1 - \alpha\theta_2$ vary in a simple harmonic way with periods $2\pi\{l(1+\alpha)/g\}^{\frac{1}{2}}$ and $2\pi\{l(1-\alpha)/g\}^{\frac{1}{2}}$ respectively, where $\alpha = (1 + \frac{m_1}{m_2})^{-\frac{1}{2}}$. Show that if the system is started from rest in the vertical position by giving B a horizontal velocity v, at time t later

$$\theta_1 + \alpha\theta_2 = \alpha v \left(\frac{1+\alpha}{gl}\right)^{\frac{1}{2}} \sin\left(t\left(\frac{g}{l(1+\alpha)}\right)^{\frac{1}{2}}\right),$$
$$\theta_1 - \alpha\theta_2 = -\alpha v \left(\frac{1-\alpha}{gl}\right)^{\frac{1}{2}} \sin\left(t\left(\frac{g}{l(1-\alpha)}\right)^{\frac{1}{2}}\right).$$

Question (1957 STEP II Q208)

A bead of unit mass slides on a rough wire in the form of a circle of radius a whose plane is vertical; θ is the angle between the radius to the bead and the downward vertical. Prove that

$$a\ddot{\theta} = -g(\mu\cos\theta + \sin\theta) - \mu a\dot{\theta}^2$$

when $0 < \theta < \frac{1}{2}\pi$ and $\dot{\theta} > 0$, μ being the coefficient of friction between bead and wire. Given that $\mu = \frac{1}{2}$ and that the motion starts with $\dot{\theta} = \omega$ at $\theta = 0$, find $\dot{\theta}$ as a function of θ during the subsequent upward motion. Find also the rate of dissipation of energy at the start. In what way, if any, is the above equation changed in the cases

1. (i) $\theta \geq 0 \geq -\frac{1}{2}\pi, \dot{\theta} > 0$;

2. (ii) $\frac{1}{2}\pi < \theta < \pi, \dot{\theta} > 0$;

3. (iii) $0 < \theta < \pi, \dot{\theta} < 0$?

Question (1957 STEP II Q209)

Two particles, A and B, of mass m and $2m$ respectively, are connected by a light rod of length $3a$; A is held fixed and B hangs in equilibrium. At $t = 0$, A is projected horizontally with velocity V. Investigate the subsequent motion of the system. Find the depth of the particles below the original position of A when AB first becomes horizontal, and determine the kinetic and potential energies of the system at that instant, taking the value of the potential energy at $t = 0$ as standard. Explain qualitatively how the motion would be altered if the rod were replaced by a light spring that remained straight during the motion, and obtain an equation determining the maximum extension of the spring during the motion in terms of its modulus of elasticity.

Question (1957 STEP II Q210)

A light inelastic string AB is suspended over a perfectly rough uniform pulley whose moment of inertia is I and radius r. The string carries at A a mass m, and at B a light spring BC of unstretched length a and elastic modulus λ. At C is attached a second mass m, and initially the system is at rest. The mass m at C is then struck with impulse P vertically downwards. Investigate the subsequent motion of the two masses.

Question (1954 STEP II Q309)

A bead of mass m slides on a smooth wire in the form of the parabola $x^2 = 4ay$, which is fixed with its axis vertical and vertex downwards. The bead is released from rest. Prove that during the subsequent motion the horizontal displacement x of the bead from the axis satisfies the equation

$$\frac{1}{4a^2}\dot{x}^2(4a^2 + x^2) = ag(c^2 - x^2),$$

where c is the initial value of x. Find the reaction of the wire on the bead when the bead is at the lowest point of the wire. If c is small, what is the period of the resulting small oscillation?

Question (1950 STEP III Q101)

A flat strip of wood, of mass M, lies on a smooth horizontal table; a particle, of mass m, rests on the strip, the upper surface of which is rough, and the coefficient of (dynamical) friction between the strip and the particle is μ. If a velocity U is suddenly given to the particle so that it moves along the strip (which may be assumed not to rotate), find for how long a time the particle slips on the strip; show also that at the instant when slipping ceases the distance through which the strip has moved is

$$mMU^2/2\mu g(m + M)^2.$$

Find how far the particle slips along the strip, and verify that the loss of kinetic energy is equal to the work done against the frictional force.

Question (1951 STEP III Q108)

A particle of mass m moves in a plane under the action of a force whose components referred to rectangular axes are $(-mn^2x, -mn^2y)$, where (x, y) are the co-ordinates of the particle. Prove that the particle moves in an ellipse with period $2\pi/n$ and that

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}mn^2(x^2 + y^2)$$

remains constant. Give the interpretation of this equation. If the particle is initially projected from the point $(a, 0)$ with the velocity (u, v), show that the axes of the ellipse lie along the bisectors of the co-ordinate axes provided $v^2 - u^2 = a^2n^2$.

Question (1954 STEP III Q108)

Apply the principles of the conservation of energy and angular momentum to solve the following problem: A light inextensible string AB passes through a small hole O in a smooth horizontal table and has particles of equal masses fastened at its two ends. Initially the mass at A on the table is held at rest and the mass at B hangs at rest. If the length of the horizontal portion of the string OA is initially r_0 and the mass at A is projected horizontally at right angles to OA with velocity $(2gr_0)^{\frac{1}{2}}$, find the length of string on the table when B is next instantaneously at rest.

Question (1955 STEP III Q210)

A smooth hollow right circular cone of semi-angle $45°$ is fixed with its axis vertical and its vertex O pointing downwards. A light elastic string of natural length a and modulus λmg passes through a small hole in the cone at O. One end of the string is fixed at a point distant a vertically below O and the other end is attached to a particle of mass m which travels on the inner surface of the cone. Initially the particle is projected horizontally with velocity $4\sqrt{(ag/3)}$ in a direction tangential to the surface of the cone at a vertical height a above O. If, in the subsequent motion, v is the component of velocity of the particle along a generator of the cone and y is the height of the particle above O, show from the equations of energy and angular momentum that

$$\frac{v^2}{2g} = 4a - \frac{8a^3}{3y^2} - y - \frac{y^2}{3a}.$$

Deduce that the height of the particle above O is always between a and $2a$.

Question (1957 STEP III Q210)

A particle of mass m can move on a smooth horizontal table and is attached to one end of a light inextensible string that passes freely through a small hole in the table and supports a particle of mass $3m$ at its other end. Initially the system is at rest with a length a of the string lying straight on the table when the particle of mass m is projected at right angles to the string with speed V. Show that when the mass $3m$ has descended a distance $\frac{1}{2}a$, assuming V to be such that this is possible, its speed will be

$$\left\{ \tfrac{1}{2}(ga - V^2) \right\}^{\frac{1}{2}},$$

and find also the tension in the string at this instant.

Question (1954 STEP III Q409)

A particle moves under an attraction varying inversely as the square of the distance from a fixed centre, and is describing a circle with period T. Show that, if it is suddenly stopped and then allowed to fall freely, it will reach the centre of force after a time $T/4\sqrt{2}$.

Question (1955 STEP III Q410)

A thin circular hoop of radius a is made of non-uniform material so that the centre of mass is halfway between centre and circumference. The hoop moves in a fixed vertical plane and can roll without slipping on a rough horizontal plane. When the centre of mass is at the highest point the hoop is slightly disturbed from rest. Find an equation giving the value θ of the angular velocity in terms of θ the angular displacement of the radius through the centre of mass from its original position. Verify that the frictional force is zero when $\cos\theta = -\frac{1}{4}$.

Question (1956 STEP III Q408)

Obtain expressions for the radial and transverse components of acceleration of a point moving in a plane and referred to polar coordinates (r, θ). A particle of mass m_1 is in motion on a smooth table and is attached by a light inextensible string, which passes freely through a small hole in the table, to a mass m_2 that hangs vertically. Show that when the distance of m_1 from the hole is r, the tension in the string is

$$\frac{m_1 m_2}{m_1 + m_2}\left(g + \frac{h^2}{r^3}\right),$$

where g is the acceleration of gravity and h is a certain constant.

Question (1948 STEP IIII Q110)

Two particles each of mass m, moving in a plane, attract each other with a force of magnitude λr^{-2}, where λ is constant and r is the distance between the particles. Prove that (i) the (linear) momentum of the system is constant in magnitude and direction, (ii) the angular momentum about the centre of mass is constant, (iii) the sum of the kinetic energy of the particles and the potential energy is constant. If initially $r = r_0$, one particle is at rest, and the other is moving with velocity v_0 at right angles to the join of the particles, show that the stationary value of r other than the value r_0 is given by

$$m v_0^2 r_0^2 \left(\frac{1}{r} + \frac{1}{r_0}\right) = \lambda.$$

Question (1917 STEP I Q108)

The tractive effort of an electric train is uniform and equal to the weight of 4 tons. The road resistance is 40 lbs. wt. per ton of the train, and the brake resistance is an additional 200 lbs. wt. per ton. The train is taken from one station to the next, distant half a mile, in $1\frac{1}{4}$ minutes, full power being kept on until the speed reaches 30 miles an hour, when the train "coasts" at a uniform speed until power is shut off and the brakes are put on. Shew that the mass of the train is approximately 85 tons.

Question (1938 STEP I Q107)

The barrel of a gun of mass M is horizontal and of length l; whilst a shell of mass m is being discharged from the gun the propelling gases exert a constant force P on the shell, and this force ceases as soon as the shell leaves the gun. From the instant of firing until the gun is brought to rest, recoil is resisted by a constant damping force $R(< P)$. Show that the total distance through which the gun recoils is

$$\frac{lmP(P - R)}{R(Pm + PM - Rm)}.$$