

**Question (1984 STEP I Q12)**

The function  $f(x)$  has first and second derivatives for all values of  $x$  and satisfies the equation

$$xf''(x) + f'(x) + xf(x) = 0,$$

together with the condition  $f(a) = 0$  for some  $a > 0$ . By considering the derivatives with respect to  $x$  of  $(xf(x)f'(x))$  and  $(x^2f'(x)^2)$ , or otherwise, show that

$$\int_0^a xf(x)^2 dx = \int_0^a xf'(x)^2 dx = \frac{1}{2}a^2[f'(a)]^2.$$

**Question (1971 STEP II Q5)**

Verify that

$$\frac{1}{2}\{f(n) + f(n+1)\} - \int_n^{n+1} f(x) dx = \frac{1}{2} \int_0^1 t(1-t)f''(t+n) dt.$$

Using the inequality

$$0 \leq t(1-t) \leq \frac{1}{4} \quad \text{if } 0 \leq t \leq 1,$$

show that

$$\frac{1}{2}\{\log n + \log(n+1)\} = \int_n^{n+1} \log x dx - r_n \quad (n > 0),$$

where

$$0 \leq r_n \leq \frac{1}{8} \left( \frac{1}{n} - \frac{1}{n+1} \right).$$

Deduce that, for all positive integers  $N$ ,

$$\log N! = \left( N + \frac{1}{2} \right) \log N - N + 1 - R_N,$$

where

$$0 \leq R_N \leq \frac{1}{8} \left( 1 - \frac{1}{N} \right).$$

**Question (1976 STEP II Q4)**

Find the straight line which gives the best fit to  $x \cos x$  for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ ; i.e., find constants  $a, b$  such that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x \cos x - ax - b)^2 dx$$

is as small as possible.

**Question (1983 STEP II Q2)**

By evaluating the integral, sketch

$$f(x) = \int_0^\pi \frac{\sin \theta d\theta}{(1 - 2x \cos \theta + x^2)^{1/2}}$$

**Question (1978 STEP III Q8)**

A function  $f(x)$  is defined, for  $x > 0$ , by

$$f(x) = \int_{-1}^1 \frac{dt}{\sqrt{(1-2xt+x^2)}}.$$

Prove that, if  $0 \leq x \leq 1$ , then  $f(x) = 2$ . What is the value of  $f(x)$  if  $x > 1$ ? Has  $f(x)$  a derivative at  $x = 1$ ?

**Question (1969 STEP III Q5)**

Evaluate

$$\int_0^\infty \int_0^\infty e^{-x} \frac{\sin cx}{x} dx dc,$$

and hence evaluate

$$\int_0^\infty \frac{\sin x}{x} dx.$$

**Question (1982 STEP III Q1)**

Find the derivative of  $\tan^{-1}[(b^2 - x^2)^{1/2}/(x^2 - a^2)^{1/2}]$  and hence evaluate

$$\int_a^b \frac{x dx}{(b^2 - x^2)^{1/2}(x^2 - a^2)^{1/2}}$$

An unknown function  $f(x)$  is related to a known continuous function  $g(z)$  by

$$g(z) = \int_0^z \frac{f(\eta) d\eta}{(z^2 - \eta^2)^{1/2}}$$

Show that the function  $f(x)$  may be found from

$$f(x) = \frac{2}{\pi} \frac{d}{dx} \int_0^x \frac{g(z)z dz}{(x^2 - z^2)^{1/2}}$$

**Question (1959 STEP III Q107)**

Let

$$f(y) = \int_{-1}^1 \frac{dx}{2\sqrt{(1-2xy+y^2)}},$$

where the positive value of the square root is taken. Prove that  $f(y) = 1$  if  $|y| \leq 1$ . Find the value of  $f(y)$  when  $|y| > 1$ . Hence or otherwise prove that if  $|y| < 1$ , then

$$\int_y^1 \frac{(x-y)dx}{(1-2xy+y^2)^{3/2}} = \int_{-y}^1 \frac{(x+y)dx}{(1+2xy+y^2)^{3/2}}.$$

**Question (1963 STEP III Q107)**

Criticize the following arguments: (i)  $\int \frac{d\theta}{5+4\cos\theta} = \int \frac{\sec^2 \frac{1}{2}\theta d\theta}{9+\tan^2 \frac{1}{2}\theta} = \frac{2}{3} \tan^{-1}(\frac{1}{3} \tan \frac{1}{2}\theta)$ ,  
 $\therefore \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{2}{3}(\tan^{-1} 0 - \tan^{-1} 0) = 0$ . (ii) The differential equation  $y'' + 2y'y = 0$  is satisfied by the functions  $y = 1/x$  and  $y = \cot x$ ; its general solution is therefore  $A \cot x + B/x$ . Another solution is  $y = \tanh x$ , therefore  $\tanh x$  is equal to a linear combination of  $\cot x$  and  $1/x$ . Solve the differential equation completely.

**Question (1964 STEP III Q309)**

Show that the function

$$f(x) = \int_x^{2x} \frac{\sin t}{t} dt$$

is bounded for  $x > 0$ , and find the points  $x$  at which it attains its greatest and least values in this range. (A function  $f(x)$  is said to be bounded over a certain range if a real number  $C$  can be found such that  $|f(x)| \leq C$  for all  $x$  in that range.)

**Question (1960 STEP II Q106)**

Obtain indefinite integrals of the functions

(i)  $\frac{x^2}{1-x}$ ,

(ii)  $\frac{\tan^3 x}{1-\tan x}$ ,

with respect to  $x$ . The following ranges of integration are proposed for the two functions respectively

(i)  $(-2, 0)$ ,  $(0, 2)$ ;

(ii)  $(0, \frac{1}{4}\pi)$ ,  $(\frac{1}{4}\pi, \frac{1}{3}\pi)$ ,  $(\frac{2}{3}\pi, \pi)$ .

State in each of the five cases whether the integration is possible, justifying your answers and giving values when they exist.

**Question (1956 STEP III Q306)**

Find

(i)  $\int_0^1 \cos^{-1} \sqrt{1-x^2} dx$ ,

(ii)  $\int_0^1 \frac{dx}{1+x^2+x^4}$ ,

(iii)  $\int \frac{dx}{(x^2+1)^{\frac{1}{2}}+(x^2-1)^{\frac{1}{2}}}$ .

**Question (1955 STEP II Q204)**

Defining an infinite integral by the equation  $\int_0^\infty f(x)dx = \lim_{X \rightarrow \infty} \int_0^X f(x)dx$ , show how to integrate an infinite integral by parts. By integration by parts, show that

$$\frac{4}{3} \int_0^\infty \frac{\sin^3 x}{x^3} dx = \int_0^\infty \frac{\sin^2 x}{x^2} dx = \int_0^\infty \frac{\sin x}{x} dx.$$

(It may be assumed that these integrals exist.)

**Question (1946 STEP III Q106)**

Prove Simpson's formula  $\frac{1}{3}h(y_0 + 4y_1 + y_2)$  for the area bounded by a curve of the type  $y = A + Bx + Cx^2$ , two ordinates of heights  $y_0, y_2$  and the axis  $y = 0$ , where  $y_1$  is the height of the mid-ordinate and  $h$  is the interval between successive ordinates. To approximate to the area under a curve for which  $y_0 = 0$  and the tangent at this point of intersection with  $y = 0$  is perpendicular to  $y = 0$ , it is sometimes convenient to fit a curve of the type  $y^2 = x^2(a + bx)$  to the points  $(0, 0), (h, y_1), (2h, y_2)$ . Show that the corresponding formula for the area is  $\frac{2}{15}h(4\sqrt{2}y_1 + y_2)$ . Illustrate these rules by finding approximately the area of a quadrant of a circle of radius  $a$ . The area is to be divided into strips of breadth  $\frac{1}{4}a$  by lines parallel to a bounding radius; for the two longer strips use Simpson's rule.

**Question (1944 STEP III Q306)**

Evaluate the integrals

$$\int_0^1 \frac{\sin^{-1} x}{(1+x)^2} dx, \quad \int_0^{\pi/2} \frac{dx}{2 \cos^2 x + 2 \cos x \sin x + \sin^2 x}, \quad \int_{-\infty}^{\infty} \frac{dx}{(e^x - a + 1)(1 + e^{-x})}.$$

**Question (1946 STEP III Q307)**

Find

$$\int \frac{(x-1)dx}{x\sqrt{1+x^2}}, \quad \int xe^x \sin x dx.$$

Prove that

$$\int_0^{\frac{\pi}{2}} \log(2 \sin x) dx = \int_0^{\frac{\pi}{2}} \log(2 \cos x) dx = \int_{\frac{\pi}{2}}^{\pi} \log(2 \sin x) dx,$$

and that each equals 0.

**Question (1944 STEP II Q107)**

Evaluate  $\int_1^\infty \frac{dx}{(1+x)\sqrt[3]{x}}$ ,  $\int_0^{2\pi} |1 + 2 \cos x| dx$ ,  $\int_2^5 \frac{x dx}{\sqrt{\{(5-x)(x-2)\}}}$ .

**Question (1944 STEP II Q110)**

State, without proof, the conditions that the expression  $A\lambda^2 + 2H\lambda + B$  should be positive for all real values of  $\lambda$ . If  $f(t)$  and  $g(t)$  are real continuous functions, show, by expressing  $\int_a^b [\lambda f(t) + g(t)]^2 dt$  in the form  $A\lambda^2 + 2H\lambda + B$ , that

$$\left[ \int_a^b f(t)g(t) dt \right]^2 \leq \int_a^b [f(t)]^2 dt \cdot \int_a^b [g(t)]^2 dt.$$

State under what conditions the equality holds. Prove that, if  $x > 0$ , then

$$e^x - 1 < \int_0^x \sqrt{(e^{2t} + e^{-t})} dt < \sqrt{\left\{ \frac{1}{2}(e^x - 1)(e^{2x} - \frac{1}{2}) \right\}}.$$

**Question (1945 STEP II Q105)**

Find

$$\int_0^\infty \frac{x dx}{x^5 + x^2 + x + 1}, \quad \int \frac{dx}{(x^3 - 1)^{\frac{1}{3}}}, \quad \int x^3 \sin x^2 dx.$$

**Question (1946 STEP II Q108)**

A function  $f(x)$  is defined, for  $x \geq 0$ , by

$$f(x) = \int_{-1}^1 \frac{dt}{\sqrt{\{1 - 2xt + x^2\}}},$$

where the positive value of the square root is to be taken. Prove that, if  $0 \leq x \leq 1$ ,  $f(x) = 2$ . What is the value of  $f(x)$  if  $x > 1$ ? Has the function  $f(x)$  a differential coefficient for  $x = 1$ ?

**Question (1944 STEP II Q408)**

Prove that:

$$(i) \quad 2\pi^3 3^{-\frac{1}{2}} > \int_0^{\pi/3} \sin^{\frac{1}{2}} x dx > 2^{\frac{1}{2}} \pi 3^{-1},$$

$$(ii) \quad 2\pi > 12 \int_0^{\pi/4} \tan^{\frac{1}{2}} x dx > \pi^{\frac{3}{2}}.$$

**Question (1946 STEP II Q406)**

(i) Prove that

$$\int_0^{\infty} \frac{dx}{1+x^3} = \int_0^{\infty} \frac{xdx}{1+x^3} = \frac{2\pi}{3\sqrt{3}}.$$

(ii) By means of the substitution  $(1 + e \cos \phi)(1 - e \cos \psi) = 1 - e^2$ , or otherwise, show that, if  $e < 1$ ,

$$(1 - e^2)^{-n-\frac{1}{2}} \int_0^{\pi} (1 + e \cos \phi)^{-n} d\phi = \int_0^{\pi} (1 - e \cos \psi)^{n-1} d\psi.$$

Hence evaluate

$$\int_0^{\pi} \frac{\sin^2 \theta d\theta}{1 + e \cos \theta}.$$

**Question (1946 STEP II Q204)**If  $y^2 = p(x - \alpha)^2 + q(x - \beta)^2$ ,  $X = r(x - \alpha)^2 + s(x - \beta)^2$ , where  $\alpha, \beta$  are unequal, prove that the substitution  $\xi = (x - \alpha)/(x - \beta)$  reduces the integral  $\int \frac{dx}{X^{n+1}y}$  to the form

$$k \int \frac{(1 - \xi)^{2n+1}}{(r\xi^2 + s)^{n+1}\eta} d\xi,$$

where  $\eta^2 = p\xi^2 + q$ , and  $k$  is a constant (to be determined). Prove that this last integral can be expressed as the sum of integrals of the types

$$(i) \int \frac{d\xi}{(r\xi^2 + s)^{m+1}\eta} \quad \text{and} \quad (ii) \int \frac{d\eta}{(r\eta^2)^{m+1}},$$

and that (i) can be found when  $m = 0$  by the substitution  $u = \eta/\xi$ .**Question (1914 STEP I Q113)**

Shew how to integrate

$$\frac{1}{(x - x_0)\sqrt{(ax^2 + 2bx + c)}},$$

and prove that the integral will be algebraical if and only if  $ax_0^2 + 2bx_0 + c = 0$ .**Question (1915 STEP I Q102)**

A sphere is divided by two parallel planes into three portions of equal volume; find to three places of decimals the ratio of the thickness of the middle portion to the diameter of the sphere.

**Question (1921 STEP I Q112)**

Evaluate the integrals

$$\int_0^1 \sqrt{\frac{1+x}{1-x}} dx, \quad \int \frac{2x^2 - 2x - 5}{2x^2 - 5x - 3} dx, \quad \int_0^{\pi} \sin^5 x dx, \quad \int x \sin x dx.$$

**Question (1926 STEP I Q114)**

Integrate

$$\int \tan^3 \theta d\theta, \quad \int \frac{dx}{x^4 + 1}, \quad \int \frac{d\theta}{a \sin \theta + b \cos \theta}.$$

**Question (1930 STEP I Q105)**

Evaluate

$$\int_0^1 \sqrt{\frac{1-x}{1+x}} dx, \quad \int_0^\infty \frac{dx}{x^4 + 1}, \quad \int_0^\infty \frac{dx}{\cosh x + 1}.$$

**Question (1931 STEP I Q109)**

Find

$$(i) \int \frac{dx}{x^2 \sqrt{x^2 + 1}}, \quad (ii) \int_0^\infty \frac{x dx}{(1 + x^2)^2};$$

and shew that, if  $a$  and  $b$  are positive,

$$\int_0^\pi \frac{\sin^2 x dx}{a^2 - 2ab \cos x + b^2} = \frac{\pi}{2a^2} \quad \text{or} \quad \frac{\pi}{2b^2},$$

according as  $a$  is greater or less than  $b$ .**Question (1932 STEP I Q106)**

Find

$$(i) \int \cot^3 x \sin^5 x dx,$$

$$(ii) \int \frac{dx}{x^4 + x},$$

$$(iii) \int_0^\infty \frac{dx}{(x+1)^2 \sqrt{x^2 + 1}}.$$

**Question (1933 STEP I Q106)**

Prove that, if  $0 < \alpha < \pi$ , then

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{1 + \cos \alpha \cos \theta} = \frac{\alpha}{\sin \alpha}.$$

What is the value of the integral when  $\pi < \alpha < 2\pi$ ? If

$$I_p = \int (x^2 + a)^p dx,$$

shew that

$$(2p + 1)I_p - 2paI_{p-1} = x(x^2 + a)^p,$$

and, hence or otherwise, evaluate

$$\int_0^{\infty} \frac{dx}{(x^2 + 1)^4}.$$

**Question (1934 STEP I Q109)**

Evaluate

$$\int \frac{dx}{\tan x + c}.$$

Shew that

$$\int_0^{\pi} \frac{(x-1)^4}{(x+1)^5} dx = \frac{1}{5} \quad \text{and} \quad \int_0^{\pi/2} \sqrt{\tan x} dx = \frac{\pi}{\sqrt{2}}.$$

**Question (1935 STEP I Q109)**

Evaluate the indefinite integrals

$$\int \frac{dx}{x(x^4 - 1)^2}, \quad \int x e^x \sin x dx, \quad \int \frac{dx}{(x-1)\sqrt{x^2 - 1}}.$$

**Question (1938 STEP I Q110)**

Evaluate the integrals

$$\int_1^2 \{\sqrt{(2-x)(x-1)}\} dx, \quad \int_0^{\infty} (1+x^2)^2 e^{-x} dx.$$

**Question (1939 STEP I Q110)**

Evaluate:

$$\int \frac{(x+1)dx}{x\sqrt{(x^2-4)}}, \quad \int_0^{\infty} \frac{dx}{\cosh^3 x}, \quad \int_0^{\pi/2} \frac{\cos x + 2 \sin x + 2}{(1 + 2 \cos x)^2} dx.$$

**Question (1940 STEP I Q110)**

Evaluate

$$\int_0^1 \sqrt{\frac{1-x}{1+x}} dx, \quad \int_0^{\pi/4} \frac{x}{\cos^4 x} dx, \quad \int_0^\infty \frac{x dx}{x^3 + 1}.$$

**Question (1942 STEP I Q103)**Find the sum of the first  $n$  terms of the series

$$\frac{1}{(1-x)(1-x^2)} + \frac{x}{(1-x^2)(1-x^3)} + \frac{x^2}{(1-x^3)(1-x^4)} + \dots$$

Deduce the sum to infinity in the cases  $|x| < 1$  and  $|x| > 1$ . Hence, or otherwise, obtain the sum of the infinite series

$$\frac{1}{\sinh u \sinh 2u} + \frac{1}{\sinh 2u \sinh 3u} + \frac{1}{\sinh 3u \sinh 4u} + \dots,$$

where  $u$  is real and different from zero.**Question (1915 STEP I Q112)**

A segment of a circle is to have a given area, and the length of the chord of the segment together with  $n$  times the length of the arc is to be a minimum. Prove that if  $n > 1$  the segment must be greater than a semicircle, and that the angle in the segment must have its secant equal to  $n$ .

What is the solution if  $n < 1$ ?**Question (1915 STEP I Q115)**

Evaluate

$$\int \frac{x dx}{(x^2 - a^2)^2 + b^2 x^2}, \quad a > 0, b > 0,$$

distinguishing between the cases in which  $b < 2a$  and  $b > 2a$ .**Question (1921 STEP I Q112)**

Evaluate  $\int_0^2 \frac{dx}{(3-x)\sqrt{2x^2+4x+9}}$ , the positive value of the root being taken. Indicate how you would proceed to evaluate the integral if  $3-x$  were replaced by  $(3-x)^2$ .

**Question (1914 STEP I Q116)**

Calculate

$$\int (x \cos x)^2 dx, \quad \int x \log x dx, \quad \int_0^\pi \frac{dx}{13 + 5 \cos x}.$$

**Question (1937 STEP I Q104)**

A variable point  $P$  lies in a fixed plane containing a fixed point  $A$ . A particle at  $P$  is under the action of a force of magnitude  $\lambda/AP$  (where  $\lambda$  is a constant) directed along  $\overrightarrow{AP}$ . Prove that, if the particle is displaced along a straight line from  $P_1$  to  $P_2$ , the work done by the force in the displacement is  $\lambda \log \frac{AP_2}{AP_1}$ . If  $B$  is another fixed point of the plane, and an additional force of magnitude  $\lambda/BP$  directed along  $\overrightarrow{BP}$  acts on the particle, prove that the work done by the resultant force in the displacement from  $P_1$  to  $P_2$  is  $\lambda \log \frac{AP_2 \cdot BP_1}{AP_1 \cdot BP_2}$ .

**Question (1915 STEP I Q114)**

Integrate the functions

$$\frac{1}{x(x^2 + a^2)}, \quad x^2 \sin^2 x, \quad e^x \cos 2x.$$

Prove that

$$\int_0^1 x^2 \log x dx = -\frac{1}{9}.$$

**Question (1916 STEP I Q112)**

If  $z = (1 - 2ax + a^2)^{-\frac{1}{2}}$ , prove that

$$\frac{\partial}{\partial x} \left\{ (1 - x^2) \frac{\partial z}{\partial x} \right\} + \frac{\partial}{\partial a} \left\{ a^2 \frac{\partial z}{\partial a} \right\} = 0.$$

**Question (1916 STEP I Q117)**

Integrate  $\int (1 + x^2)e^x dx$ ;  $\int \sec^3 x dx$ ; and prove that

$$\int_0^\infty \frac{(3x + 4)dx}{(x + 2)(x^2 + x + 1)} = \frac{2}{3} \log_e 2 + \frac{4\pi}{3\sqrt{3}}.$$

**Question (1917 STEP I Q115)**

Find  $\int \sin^{-1} x dx$ ,  $\int \frac{\sin^2 x dx}{1 + \cos^2 x}$ ,  $\int_0^\infty \frac{dx}{(1+x^2)^2}$ ,  $\int_0^a \frac{x^3 dx}{\sqrt{a-x}}$ .

**Question (1919 STEP I Q114)**

Evaluate the integrals

$$\int \frac{x^2 + 2x + 2}{(x + 1)^2} dx, \quad \int x \sin x dx, \quad \int_{-1}^1 (1 - x^2)^{\frac{3}{2}} dx.$$

**Question (1921 STEP I Q103)**

The infinite series

$$c_0 + c_1 + \cdots + c_n + \cdots \quad (1)$$

and the infinite continued fraction

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots + \frac{a_n}{b_n + \cdots}}} \quad (2)$$

are said to be equivalent if, for each value of  $n$ ,

$$S_n = \frac{p_n}{q_n},$$

where

$$S_n = c_0 + c_1 + \cdots + c_n,$$

$$\frac{p_n}{q_n} = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots + \frac{a_n}{b_n}}}$$

By finding the value of  $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}$ , or otherwise, shew that the continued fraction (2) is equivalent to the series

$$b_0 + \frac{a_1}{q_1} - \frac{a_1 a_2}{q_1 q_2} + \cdots + (-1)^{n-1} \frac{a_1 a_2 \cdots a_n}{q_{n-1} q_n} + \cdots$$

Again, by solving the equations

$$s_n = b_n s_{n-1} + a_n s_{n-2},$$

$$1 = b_n \cdot 1 + a_n \cdot 1,$$

for  $a_n$  and  $b_n$ , or otherwise, shew that the series (1) is equivalent to the continued fraction

$$c_0 + \frac{c_1}{1 - \frac{c_2/c_1}{1 + c_2/c_1 - \frac{c_3/c_2}{1 + c_3/c_2 - \cdots - \frac{c_n/c_{n-1}}{1 + c_n/c_{n-1} - \cdots}}}}$$

and find the series equivalent to

$$a_0 + \frac{\alpha_1}{1 - \frac{\alpha_2}{1 + \alpha_2 - \cdots - \frac{\alpha_n}{1 + \alpha_n - \cdots}}}$$

**Question (1924 STEP I Q105)**

Explain the usual process for finding the H.C.F. of two polynomials  $U(x), V(x)$  and shew that, if they have no common factor, then polynomials  $L(x), M(x)$  can be found such that

$$L(x)U(x) + M(x)V(x) = W(x),$$

where  $W(x)$  is any other given polynomial. Discuss the case

$$U(x) = Q(x), \quad V(x) = \frac{d}{dx}Q(x), \quad W(x) = P(x),$$

where  $Q(x)$  has no repeated factors and the degree of  $P(x)$  is less than that of  $Q(x)$ , and shew how a reduction formula for the integral

$$\int \frac{P(x)}{[Q(x)]^n} dx$$

may be found. Reduce, and thus evaluate

$$\int \frac{2x^3 - 1}{(x^3 - 3x + 1)^2} dx.$$

**Question (1925 STEP I Q105)**

The functions  $f$  and  $\phi$  are supposed to have as many derivatives as may be required over the ranges considered. Shew that, if  $\phi$  is subject to the conditions

$$\phi'(0) = 0, \quad \phi(h) = \phi'(h) = \phi''(h) = 0, \quad \dots\dots\dots (1)$$

then

$$\int_0^h F(t)\phi'''(t)dt = F(h)\phi'''(h) - F(0)\phi'''(0) + \int_0^h F''(t)\phi''(t)dt,$$

where

$$F(t) = f(c+t) + f(c-t),$$

$c$  and  $h$  being real constants and  $h$  positive. Shew that there is a polynomial in  $t$  of the fourth degree satisfying (1), and deduce the formula

$$\int_{c-h}^{c+h} f(x)dx = \frac{h}{3}[f(c-h)+4f(c)+f(c+h)] - \frac{1}{72} \int_0^h (h-t)^3(h+3t)\{f''''(c+t)+f''''(c-t)\}dt.$$

Deduce that, if  $f$  is a polynomial of the third or lower degree, the value ( $\alpha$ ) of the integral on the left is given exactly by the first term ( $\alpha'$ ) on the right, and that if  $f$  is a general function whose fourth derivative satisfies the inequalities

$$l < f''''(x) < L, \quad (c-h \leq x \leq c+h) \dots\dots (2)$$

then limits to the error involved in taking  $\alpha'$  as an approximation to  $\alpha$  are given by

$$\frac{1}{90}h^5l \leq \alpha' - \alpha \leq \frac{1}{90}h^5L.$$

Deduce that, if we approximate to the definite integral

$$A = \int_a^b f(x)dx, \quad (b > a)$$

by Simpson's rule, i.e. by dividing  $(a, b)$  into  $2n$  parts each of length  $h$  ( $2nh = b - a$ ), by points

$$a = x_0, x_1, \dots, x_{2n} = b,$$

and taking as an approximation the sum

$$A' = \frac{h}{3}[y_0 + y_{2n} + 2(y_2 + y_4 + \dots + y_{2n-2}) + 4(y_1 + y_3 + \dots + y_{2n-1})],$$

where  $y_r = f(x_r)$ , then limits to the error will be given by

$$\frac{1}{180}(b-a)h^4l \leq A' - A \leq \frac{1}{180}(b-a)h^4L,$$

provided that (2) is satisfied throughout the interval  $a \leq x \leq b$ .

**Question (1931 STEP I Q104)**

Prove that the radius of curvature  $\rho$  of a curve  $f(x, y) = 0$  is given by the formula

$$\frac{1}{\rho} = (f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2)/(f_x^2 + f_y^2)^{3/2},$$

where suffixes denote partial differentiation. The equation  $f(x, y, a) = 0$  represents a family of curves. Prove that, if  $\rho, \rho'$  denote the radii of curvature of a particular curve and of the envelope of the family at the point where they touch, then

$$f_{aa} \left( \frac{1}{\rho} - \frac{1}{\rho'} \right) + (f_y f_{ax} - f_x f_{ya})(f_x^2 + f_y^2)^{-3/2} = 0.$$

**Question (1940 STEP I Q106)**

If  $m$  and  $n$  are unequal integers, prove that

$$\int_X^Y \frac{\sin^2 \pi x}{x(x-m)(x-n)} dx = \frac{1}{m-n} \left( \int_{Y-n}^{Y-m} \frac{\sin^2 \pi u}{u} du - \int_{X-n}^{X-m} \frac{\sin^2 \pi u}{u} du \right)$$

and hence, or otherwise, find the value of

$$\int_{-\infty}^{\infty} \frac{\sin^2 \pi x}{x(x-m)(x-n)} dx.$$

**Question (1916 STEP I Q105)**

The circle of curvature of a curve, at a point  $P$ , may be defined (1) as a circle which passes through  $P$  and has its centre at the limiting position of the intersection of the normals at  $P$  and a neighbouring point  $Q$ ; (2) as the limit of a circle which touches the curve at  $P$  and passes through a neighbouring point  $Q$ ; (3) as the limit of a circle passing through  $P$  and two neighbouring points  $Q$  and  $R$ ; (4) as the circle which has the closest possible contact with the curve at  $P$ . Give a careful proof that these four definitions are equivalent. Obtain formulae for the radius of curvature of curves given in the forms (1)  $s = f(\psi)$ , (2)  $f(x, y) = 0$ , (3)  $x = \phi(t), y = \psi(t)$ ; and apply one or other of them to the parabola, ellipse, four-cusped hypocycloid, and catenary.

**Question (1920 STEP I Q113)**

Evaluate the integrals

$$\int x^2 \log x dx, \quad \int \frac{\sqrt{(x^2 - a^2)}}{x} dx, \quad \int \frac{1}{5 + 4 \sin x} dx, \quad \int_0^{\pi/2} \cos^2 \theta \sin^2 2\theta d\theta.$$

**Question (1921 STEP II Q208)**

Prove the formula  $\rho = r \frac{dr}{dp}$  for the radius of curvature of a curve  $f(r, p) = 0$ . If  $t$  is the length of the tangent from the pole to the circle of curvature at any point of the curve, show that  $t^2 = \frac{d}{dr}(r^3 v)$ , where  $v = \frac{1}{p}$ . Deduce that if all the circles of curvature of a curve pass through a fixed point, the curve must be a circle.

**Question (1921 STEP II Q209)**

Integrate

$$\int \frac{dx}{x\sqrt{1+x^2}}, \quad \int \frac{dx}{(x+1)^2(x^2+x+1)}, \quad \int x^2 \sec^2 x \tan x \, dx.$$

If  $\int f(x) \, dx = \log[1 + f(x)]$ , determine  $f(x)$ .

**Question (1923 STEP II Q209)**

Evaluate the following:

$$\int_a^b \sqrt{(b-x)(x-a)} \, dx, \quad \int \frac{dx}{(a+b \sin x) \cos x}, \quad \int \frac{(1-x^2)dx}{(1+x^2)\sqrt{1+x^2+x^4}}.$$

**Question (1926 STEP II Q208)**

Show how to find  $\int \frac{ax^2+2bx+c}{(Ax^2+2Bx+C)^2} \, dx$ . Find the condition that it should be a rational function.

**Question (1929 STEP II Q210)**

(i) Evaluate

$$\int \frac{(x-3)dx}{4x^2+5x+1}.$$

(ii) Given  $\log_{10} e = 0.4343$ , prove that

$$\int_1^3 x \log_x \left(1 + \frac{1}{x}\right) dx = 1.601.$$

(iii) Find a reduction formula for  $\int_0^{\pi/2} \sin^n x \, dx$  where  $n$  is a positive integer, and evaluate

$$\int_0^{\pi/2} \sin^6 x \, dx.$$

**Question (1933 STEP II Q209)**

Integrate

$$x^2 \sqrt{1+x^2}, \quad \frac{\cos^2 2x}{\sin^4 x \cos^2 x}, \quad x^m (\log x)^2.$$

**Question (1933 STEP II Q210)**

(i) Evaluate

$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)(x^2 + c^2)},$$

where  $a, b$  and  $c$  are positive.

(ii) Find a reduction formula for the integral

$$\int_0^{\frac{\pi}{4}} \frac{dx}{\cos^n x},$$

and evaluate the integral for the cases  $n = 1, 2$ .**Question (1936 STEP II Q209)**A function  $f(x)$  is defined, for  $x \geq 0$ , by

$$f(x) = \int_{-1}^1 \frac{dt}{\sqrt{\{1 - 2xt + x^2\}}}.$$

Prove that, if  $0 \leq x \leq 1$ ,  $f(x) = 2$ . What is the value of  $f(x)$  if  $x > 1$ ? Has the function  $f(x)$  a differential coefficient for  $x = 1$ ?**Question (1938 STEP II Q209)**

Evaluate

1.  $\int_0^a \frac{dx}{x + \sqrt{(a^2 - x^2)}}$ ,
2.  $\int_0^{2\pi} \frac{\sin^2 \theta d\theta}{a - b \cos \theta}$ , where  $a > b > 0$ ,
3.  $\int_1^{\infty} \frac{dx}{(1+x)\sqrt{x}}$ .

**Question (1938 STEP II Q210)**

Define the area of the surface of a body formed by the revolution of a curve about a straight line in its plane. A circular arc revolves about its chord; prove that the area of the surface generated is  $4\pi a^2(\sin \alpha - \alpha \cos \alpha)$ , where  $a$  is the radius and  $2\alpha (< \pi)$  is the angular measure of the arc.

**Question (1939 STEP II Q209)**

Evaluate

- (i)  $\int_1^{\infty} \frac{dx}{x^2(a^2+x^2)^{\frac{1}{2}}}$ ,
- (ii)  $\int_0^{\pi/2} \frac{2+3\cos x}{(2+\cos x)^2} dx$ .

**Question (1940 STEP II Q210)**

(i) Prove that, if  $n$  is a positive integer,

$$\int_0^{\pi/2} e^{\lambda x} \cos nx dx = \frac{1}{\lambda^2 + n^2} \{\lambda e^{\lambda\pi/2} - 1\},$$

where  $\lambda$  has one of the values  $\pm 1, \pm n$ , and classify the cases.

(ii) Find the area bounded by the parabola  $y^2 = ax$  and the circle  $x^2 + y^2 = 2a^2$ .

**Question (1941 STEP II Q209)** 1. Find the indefinite integrals

$$\int \frac{(1+x^2) dx}{x^2(1-x)}, \quad \int \cos^4 x dx.$$

2. Evaluate the integral

$$\int_a^b x^2 \sqrt{\{(x-a)(b-x)\}} dx,$$

where  $0 < a < b$ .

**Question (1917 STEP III Q203)**

Four points lie on a circle: shew that the six perpendiculars, each drawn from the middle point of a chord joining two points to the chord joining the other two, pass through the same point the line joining which to the centre of the circle is bisected by the centroid of the four points. Prove that the point of intersection of the perpendiculars is the centre of the rectangular hyperbola through the four points and give the limiting form of the theorem when three of the four points coincide so that the circle is the circle of curvature at a point of the rectangular hyperbola.

**Question (1920 STEP III Q213)**

Integrate

(i)  $\int \frac{dx}{\sin x + \cos x},$

(ii)  $\int \frac{dx}{(x-1)^2(x+1)},$

(iii)  $\int_1^2 \sqrt{\{(x-1)(2-x)\}} dx.$

**Question (1923 STEP III Q203)**

Prove that the rectangles contained by the segments of any two intersecting chords of a conic are to one another as the squares of the parallel tangents taken between their intersection and their points of contact. Two chords of an ellipse  $Pp, Qq$  are at right angles and are normal to the ellipse at  $P$  and  $Q$  respectively: show that  $pq$  is parallel to  $PQ$ .

**Question (1921 STEP III Q204)**

The diameter of a sphere is divided into two parts (of lengths  $p, q$ ) by a perpendicular plane which divides the sphere into segments of volumes  $V, V'$  and of spherical surfaces  $S, S'$ . Find (by the Integral Calculus or otherwise) formulae for  $V, V'$  and  $S, S'$  in terms of  $p, q$ . Obtain the theorems of Archimedes

$$(i) V : V' < S^2 : S'^2,$$

$$(ii) V^2 : V'^2 > S^3 : S'^3,$$

provided that  $p$  is greater than  $q$ . Shew that if  $t = q/p$ , then

$$\frac{V'^2}{S'^3} = \frac{(1 + 3t)^2}{36\pi(1 + t)^3};$$

(Note: Transcribed from image, differs from OCR.) and prove that when  $S'$  is given,  $V'$  is greatest for a hemisphere.

**Question (1925 STEP III Q205)**

If  $ax + by + cz = 1$  and  $a, b, c$  are positive, shew that the values of  $x, y, z$  for which  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$  is stationary are given by

$$ax^2 = by^2 = cz^2.$$

Shew that this is a true maximum or minimum if  $xyz > 0$ .

**Question (1919 STEP I Q311)**

Simpson's rule for finding areas by approximation is based on the property that, if  $y_1, y_2, y_3$  are three ordinates of the parabola  $y = a + bx + cx^2$  separated by equal intervals, the mean ordinate of the portion of the curve between the ordinates  $y_1$  and  $y_3$  is  $\frac{1}{3}(y_1 + 4y_2 + y_3)$ . Prove this and deduce the rule. Find an approximate value for

$$\int_0^{10} \sqrt{6 + 5x - 3x^2 + x^3} dx.$$

**Question (1938 STEP I Q307)**

Evaluate

$$\int_0^{\infty} \frac{x^2 dx}{(1 + x^2)^{5/2}}$$

and

$$\int_{-\infty}^{\infty} \frac{dx}{(e^{x/2} + 1)(e^{-x/2} + 1)}.$$

**Question (1938 STEP I Q309)**

Prove that  $\int_0^x \frac{\sin y}{y} dy$  is positive when  $x$  is positive.

**Question (1939 STEP I Q308)**

Find the volume of the body

$$(\sqrt{x^2 + y^2} - a)^2 < b^2 - z^2$$

for  $0 < b < a$  and for  $0 < a < b$ .

**Question (1939 STEP I Q309)**

If  $0 < a < b$  and if for  $a < x < b$

$$f(x) \geq 0, \quad xf'(x) + f(x) \geq 0,$$

prove, by partial integration or otherwise, that

$$\left| \int_a^b f(x) \cos(\log x) dx \right| < 2bf(b).$$

**Question (1941 STEP I Q304)**

Evaluate the integrals

$$\int \frac{dx}{(1+x)(4+6x+4x^2+x^3)}, \quad \int \frac{\sin^2 x dx}{\cos^3 x}, \quad \int_0^{\pi/2} \log(\sin x \cos x) dx.$$

**Question (1942 STEP I Q303)**

Evaluate the integrals:

$$1. \int \frac{x-1}{x^2} e^x dx; \quad (ii) \int \frac{(2x^3+1)dx}{x(x^3+1)\sqrt{x^6(1+x^3)^2+1}}.$$

**Question (1913 STEP II Q307)**

Given  $F\{s^2(z-x), s^3(z-y)\} = 0$ , where  $s = x + y + z$ , prove that

$$(s-x) \frac{\partial z}{\partial x} + (s-y) \frac{\partial z}{\partial y} = s-z.$$

**Question (1925 STEP II Q301)**

Show that  $(ay - bx)^2 - (bz - cy)(cz - az)$  is the product of two linear factors which are real if  $c^2 > 4ab$ . If  $x + y + z + w = 0$ , prove that

$$wx(w+x)^2 + yz(w-x)^2 + wy(w+y)^2 + zx(w-y)^2 + wz(w+z)^2 + xy(w-z)^2 + 4xyzw = 0.$$

**Question (1925 STEP II Q309)**

Integrate

(i)  $\int \frac{\sqrt{x-1}}{x\sqrt{x+1}} dx,$

(ii)  $\int_0^1 \frac{dx}{(1+x)(2+x)\sqrt{x(1-x)}},$

(iii)  $\int \frac{\sin x dx}{4 \cos x + 3 \sin x},$

(iv)  $\int \frac{\sqrt{a^2 + b^2 \cos^2 x}}{\cos x} dx.$

**Question (1926 STEP II Q309)**

Integrate

(i)  $\int \sin 3x \cdot \sin 4x \cdot dx,$

(ii)  $\int \log_e(x + \sqrt{a^2 + x^2}) dx,$

(iii)  $\int \frac{x^2+2}{x^4+4} dx,$

(iv)  $\int_0^{\frac{\pi}{2}} \frac{dx}{4+5 \cos x}.$

**Question (1927 STEP II Q309)**

Integrate

(i)  $\int x \tan^{-1} x dx,$

(ii)  $\int \frac{dx}{x\sqrt{x^2+2x-3}}.$

If

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5,$$

prove that

$$\int_{-1}^1 y dx = \frac{1}{9}(8y_0 + 5y_1 + 5y_2),$$

where  $y_0, y_1, y_2$  are the values of  $y$  when  $x = 0, \sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}$  respectively.

**Question (1930 STEP II Q306)**

A chord is drawn to cut a circle of radius  $a$  so that the smaller segment is one-sixth of the total area. Shew that if the distance  $p$  of the chord from the centre be  $a \cos \theta$ ,  $\theta$  is given in radians by

$$\sin \theta \cos \theta - \theta + \frac{\pi}{6} = 0.$$

Prove that this equation has a root between  $\frac{\pi}{4}$  and  $\frac{\pi}{3}$  and (by simple interpolation or otherwise) deduce an approximate value for  $p$ .

**Question (1913 STEP III Q310)**

Interpret the expressions  $\int x \frac{dy}{ds} ds$  and  $\int y \frac{dx}{ds} ds$  when taken round the boundary of a closed plane curve. Trace the curve

$$y^4 - 2xy + x^3 = 0$$

and prove that the area of a loop is  $\frac{2}{5}$ .

**Question (1919 STEP III Q310)**

Integrate with respect to  $\theta$  the expressions  $\frac{1}{\sin^3 \theta}$  and  $\frac{5}{1+2 \cot \theta}$ . Prove that the straight line  $2a^2x = 9b^2y$  cuts off from the curve  $b^2y = x^2(a-x)$  two segments which are equal in area.

**Question (1922 STEP III Q310)**

Prove that

$$\left(\frac{d^2x}{d\phi^2}\right)^2 + \left(\frac{dy}{d\phi}\right)^2 = \frac{1}{\rho^2} \left(\frac{d\rho}{d\phi}\right)^2 + \rho^2 + 4 \left(\frac{dp}{d\phi}\right)^2.$$

$$\left(\frac{d^2x}{d\phi^2}\right)^2 + \left(\frac{dy}{d\phi}\right)^2 = \frac{1}{\rho^2} - \rho + 4 \left(\frac{dp}{d\phi}\right)^2.$$

where  $x, y$  are the coordinates of a point on a plane curve,  $\rho$  the radius of curvature at the point and  $\phi$  the angle the tangent makes with a fixed direction.

**Question (1931 STEP III Q310)**

(i) Prove that

$$\int_1^\infty \frac{dx}{x(1+x^3)} = \frac{2}{3} \log_e 2.$$

(ii) Find the area of the curve

$$a^2y^2 = x^2(a^2 - x^2).$$

**Question (1934 STEP III Q310)**

Prove that

$$(i) \int_0^3 \frac{x dx}{\sqrt{3+6x-x^2}} = \pi - \sqrt{3}.$$

$$(ii) \int_0^{\pi/3} (1 + \tan^6 \theta) d\theta = \frac{8}{5}\sqrt{3}.$$

$$(iii) \int_1^3 \frac{x^2+1}{x^4+7x^2+1} dx = \frac{1}{3} \tan^{-1} \frac{1}{2}.$$

**Question (1942 STEP III Q306)**

Evaluate

$$\int_a^b \sqrt{\{(b-x)/(x-a)\}} dx, \quad a < b$$

by means of the substitution  $x = a \sin^2 \theta + b \cos^2 \theta$  or otherwise. Make a rough drawing of the curve

$$x^3 + 3xy^2 - 3a(x^2 - y^2) = 0,$$

and shew that the area of its loop is  $3a^2$ .

**Question (1920 STEP III Q309)**

Evaluate

$$\int x \sin^{-1} x dx, \quad \int \frac{3x^2 + x - 1}{(x^2 + 1)(x + 1)^2} dx, \quad \int \frac{cx + f}{(ax^2 + 2bx + c)^3} dx.$$

Prove that, if  $a$  and  $b$  are positive, then

$$\int_0^\pi \frac{\sin^2 x dx}{a^2 + b^2 - 2ab \cos x} = \frac{\pi}{2a^2} \text{ or } \frac{\pi}{2b^2},$$

according as  $a >$  or  $< b$ .

**Question (1921 STEP III Q310)**

Explain the application of the integral calculus to the computation of areas (i) in Cartesian, (ii) in polar coordinates. P and Q are two points on a rectangular hyperbola whose centre is C and PL, QM are perpendiculars on an asymptote. Prove that the area bounded by the lines CP, CQ and the arc PQ is equal to that bounded by the lines PL, LM, MQ and the arc PQ.

**Question (1922 STEP III Q310)**

Evaluate:

(i)  $\int \frac{dx}{(x^2+a^2)^3};$

(ii)  $\int \frac{dx}{(x-2)\sqrt{x^2+2x+3}};$

(iii)  $\int \frac{\sin x dx}{4 \cos x + 3 \sin x};$

(iv)  $\int_0^{\frac{\pi}{2}} e^{ax} \sin x dx.$

**Question (1923 STEP III Q309)**

Prove the following results:

$$\int_0^\pi \frac{dx}{a + b \cos x} = \frac{\pi}{\sqrt{a^2 - b^2}} \quad (a^2 > b^2 \text{ and } a > 0).$$

$$\begin{aligned} \int_{-1}^1 \frac{\sin \alpha dx}{1 - 2x \cos \alpha + x^2} &= \frac{1}{2} \pi \quad (0 < \alpha < \pi) \\ &= -\frac{1}{2} \pi \quad (\pi < \alpha < 2\pi). \end{aligned}$$

$$\int_0^1 x^{2n-1} \log(1+x) dx = \frac{1}{2n} \left\{ \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2n-1)2n} \right\}.$$

**Question (1937 STEP III Q305)**

Evaluate

$$\int_0^\infty \frac{dx}{(x+1)\sqrt{(5x^2+12x+8)}}.$$

**Question (1916 STEP I Q410)**

Evaluate  $\int_0^\infty \frac{dx}{\sqrt{x(4-x)(x-3)}}$  and  $\int_0^\infty \frac{dx}{(2+x)\sqrt{x(1+x)}}$ . Shew without integration that  $\int_0^{\frac{2\pi}{3}} \frac{64d\theta}{(5+3\cos\theta)^2}$  lies between 644 and 753; and, by integrating, that its value is about 68. (Take  $\arctan \frac{1}{2} = .322$  and  $\arctan \frac{1}{4} = .165$ .)

**Question (1917 STEP I Q410)**

Perform the integrations:

$$\int \frac{(6x^3 + 3x) dx}{(x^2 - 1)(x - 1)}, \quad \int \frac{dx}{\sqrt[4]{\{(7-x)(x-3)\}}}, \quad \int \sqrt{(\sec x + 1)} dx.$$

Transform the last integral by writing  $\sec x = \sec^2 \frac{1}{2}y$ .

**Question (1913 STEP II Q407)**

Integrate with respect to  $x$

$$\frac{1}{1+x+x^2}, \quad \frac{1}{(x+1)\sqrt{x^2+x+2}}, \quad x \sin 2x \sin 3x.$$

By means of the substitution

$$\tan \frac{1}{2}\theta = \sqrt{\frac{1+e}{1-e}} \tan \frac{1}{2}\phi,$$

evaluate

$$\int_0^\pi \frac{d\theta}{(1+e \cos \theta)^2}.$$

**Question (1913 STEP II Q410)**

Find the area of a loop of the curve

$$(x^2 + 4y^2)^2 = x^2 - 9y^2.$$

**Question (1920 STEP II Q407)**

The radius  $R$  of the circumcircle of the triangle  $ABC$  is expressed in terms of  $a, b$  and  $C$ ; find  $\frac{\partial R}{\partial a}$  and prove that

$$\frac{\partial R}{\partial C} = R \frac{\cos A \cos B}{\sin C}.$$

**Question (1921 STEP II Q409)**

Evaluate the integrals

$$\int \frac{dx}{(2+x)\sqrt{1+x}}, \quad \int \cos x \cos 3x dx, \quad \int_0^\pi \frac{d\theta}{a+b \cos \theta} \quad (a > b).$$

**Question (1922 STEP II Q410)**

Evaluate the integrals

(i)  $\int (x+1)\sqrt{x^2+x+1} dx,$

(ii)  $\int \frac{dx}{x^4+1},$

(iii)  $\int x^2 \sin^{-1} x dx,$

(iv)  $\int \frac{dx}{\sin(x-a)\sin(x-b)}.$

**Question (1923 STEP II Q409)**

Interpret the expressions (i)  $\int x \frac{dy}{ds} ds$ , (ii)  $\int y \frac{dx}{ds} ds$ , (iii)  $\int (x \frac{dy}{ds} - y \frac{dx}{ds}) ds$ , where the integrals are taken round the perimeter of a closed curve. Find the area of a loop of the curve  $x^5 + y^5 = ax^2y^2$ .

**Question (1923 STEP II Q410)**

Find the integrals

$$\int \frac{dx}{\sqrt{2+x+x^2}}, \quad \int \frac{dx}{1+x^3}, \quad \int_{\alpha}^{\pi-\alpha} \frac{dx}{\tan x - \tan \alpha},$$

$$\int_0^{\pi} \frac{dx}{(3+2\cos x)^2}, \quad \int_0^1 x^{n-1}(1-x)^n dx.$$

**Question (1924 STEP II Q409)**

Integrate:

1.  $\int \frac{(x+1)dx}{(x+2)\sqrt{x^2+4}}$ ,
2.  $\int_0^{\pi/2} x^2 \cos x dx$ ,
3.  $\int \frac{\sin x dx}{\sin(x-a)\sin(x-b)}$ ,
4.  $\int_0^{\pi} \frac{\sin^2 x dx}{a^2 - 2ab \cos x + b^2} \quad (a > b > 0)$

**Question (1933 STEP II Q406)**

Prove that

$$\int_0^{\infty} \frac{dx}{x^2 + 2x \cos \alpha + 1} = \frac{\alpha}{\sin \alpha} \quad 0 < \alpha < \pi.$$

Evaluate  $\int_0^{\infty} \frac{dx}{x^4 + 2x^2 \cos \alpha + 1}$  and  $\int_0^{\infty} \frac{(x^2 + 1)dx}{x^4 + 2x^2 \cos \alpha + 1}$ .

**Question (1934 STEP II Q407)**

Evaluate

- (i)  $\int \sqrt[3]{\frac{a^3-x^3}{1-x^3}} x dx, \quad a > 1;$
- (ii)  $\int \log \frac{x(1+x^2)}{(1+x)(1+x^3)} dx.$

**Question (1918 STEP III Q408)**

Evaluate  $\int \sec^3 x dx$ ,  $\int \frac{3x+2}{\sqrt{\{x^2+4x+1\}}} dx$ . Prove that

$$\int_1^{\infty} \frac{x^2 + 2}{x^4(x^2 + 1)} dx = \frac{\pi}{4} - \frac{1}{3}.$$

**Question (1918 STEP III Q409)**

Prove that the area of the curved surface and the volume of a segment of height  $h$  of a sphere of radius  $a$  are  $2\pi ah$  and  $\frac{1}{3}\pi h^2(3a - h)$ . The whole area (curved and plane) of a segment of a sphere is given to be equal to  $\pi c^2$ . Prove that when the volume is greatest the height of the segment is  $c$ .

**Question (1930 STEP III Q410)**

Prove that

$$(i) \int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \frac{\pi}{2};$$

$$(ii) \int_0^{\pi} \frac{dx}{1+\cos \lambda \cos x} = \frac{\lambda}{\sin \lambda}.$$

**Question (1937 STEP III Q409)**

Find the integral

$$\int (1 - x^2)^{\frac{3}{2}} dx,$$

and evaluate

$$\int_2^3 \frac{dx}{[(x-1)(3-x)]^{\frac{3}{2}}} \quad \text{and} \quad \int_{a+b}^{a+2b} \frac{dx}{2a+b-\sqrt{(x-a)(x-b)}}.$$

**Question (1941 STEP III Q405)**

Prove for positive values of  $x$ , that if  $p > q > 0$ , then

$$q(x^p - 1) \geq p(x^q - 1).$$

Hence, or otherwise, show that for,  $x \geq 1$ , and  $p \geq q > 0$ ,

$$1. \frac{x^p-1}{p} \geq \frac{x^q-1}{q} - \frac{(p-q)}{pq} \log_e x.$$

$$2. \frac{qn}{p-n}(x^p - 1) \geq \frac{pn}{q-n}(x^q - 1) - \frac{pq(p-q)}{(p-n)(q-n)}(x^n - 1),$$

where  $p \neq n, q \neq n$ , and  $n > 0$ .

**Question (1941 STEP III Q410)** 1. Evaluate

$$\int_0^{2\pi} \frac{\cos(n-1)x - \cos nx}{1 - \cos x} dx,$$

where  $n$  is a positive integer.

2. Find the limit of  $\frac{1}{x^2} - \cot^2 x$ , as  $x \rightarrow 0$ .

**Question (1942 STEP III Q409)**

If  $0 < \theta_1 < \theta_2 < \pi$ , prove that the volume swept out in one complete revolution about the line  $\theta = 0$  by the plane region bounded by the curve  $r = f(\theta)$  and the lines  $\theta = \theta_1, \theta = \theta_2$  is

$$\frac{2\pi}{3} \int_{\theta_1}^{\theta_2} r^3 \sin \theta d\theta.$$

Prove that the locus of a point which moves so that the product of its distances from two fixed points at a distance  $2c$  apart is  $c^2$  is a surface of revolution enclosing a volume

$$\pi c^3 \left\{ \log(1 + \sqrt{2}) - \frac{\sqrt{2}}{3} \right\}.$$

**Question (1942 STEP III Q410)**

Find the values of:

- $\int_2^5 (x^2 - 7x + 15) dx$  and  $\int_3^{15} y(4y - 11)^{-\frac{1}{2}} dy$ ,
- $\int_4^5 (x^2 - 7x + 15) dx$  and  $\int_3^5 y(4y - 11)^{-\frac{1}{2}} dy$ ,

and explain why the substitution  $y = x^2 - 7x + 15$  appears to fail as a means of evaluating the first integral but to succeed in the case of the third integral. Show how to evaluate the first integral correctly by this substitution. Explain why the substitution  $y = \sin x$  in  $\int_{-\pi}^{\pi} \cos^3 x dx$  appears to give  $\int_0^0 (1 - y^2) dy = 0$ .

**Question (1933 STEP III Q402)**

If  $r$  denotes the distance of a point  $Q$  lying on a given curve from a fixed point  $S$  in the plane of the curve, and  $p$  is the perpendicular distance from  $S$  to the tangent at  $Q$  to the given curve, shew that the radius of curvature at  $Q$  is  $r \frac{dr}{dp}$ . If the given curve is an ellipse of semi-axes  $a$  and  $b$  ( $a > b$ ) and  $S$  is a focus, shew that

$$\frac{l}{p^2} = \frac{2}{r} - \frac{1}{a}, \quad \text{where } l = b^2/a,$$

and hence determine its maximum and minimum radii of curvature.

**Question (1925 STEP II Q508)**

Prove that for a plane curve  $p = r \frac{dr}{dp}$ . Prove that the radius of curvature of  $r^n = a^n \cos n\theta$  is  $a^n/(n+1)r^{n-1}$  and find the  $(p, r)$  equation of the evolute of the curve.

**Question (1925 STEP II Q510)**

Show that the area of the surface of the spheroid formed by revolving the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  about the axis of  $y$  is  $2\pi a^2 \left[ 1 + \frac{1-e^2}{e} \tanh^{-1} e \right]$ , where  $e$  is the eccentricity of the ellipse.

**Question (1926 STEP II Q510)**

Find the integrals:

$$\int \frac{dx}{(x-2)\sqrt{x^2+2x+3}}, \quad \int_0^a x^2(\log x)^2 dx, \quad \int_c^b x^2(x-a)^{\frac{1}{2}}(b-x)^{\frac{1}{2}} dx,$$

where  $c$  is a constant. Find the length of the spiral  $r = 3\theta$  from  $\theta = 0$  to  $\theta = \frac{\pi}{3}$ .

**Question (1927 STEP II Q509)**

Evaluate

$$\int \frac{dx}{\sqrt{(1+\sin x)(2-\sin x)}}.$$

Prove that

$$\int_0^1 \frac{(4x^2+3)dx}{8x^2+4x+5} = \frac{1}{4} - \frac{1}{8} \log \frac{17}{5} + \frac{1}{4} \tan^{-1} \frac{6}{7},$$

and

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+2\cos x} = \frac{1}{\sqrt{3}} \log(2+\sqrt{3}).$$

**Question (1920 STEP III Q510)**

For a curve defined by the equation  $p = f(\psi)$  prove that the projection of the radius vector on the tangent is  $\frac{dp}{d\psi}$ . For the curve

$$p = a \sin n\psi$$

prove that

$$r^2 = a^2 n^2 + (1 - n^2)p^2.$$

**Question (1932 STEP III Q509)**

Explain how to find the intrinsic  $(s, \psi)$  form of the equation of a plane curve whose pedal  $(p, r)$  equation is known. Shew that the  $(s, \psi)$  equation of the curve  $p^3 = ar^2$  is  $s = 3a \tan \frac{\psi}{2} + a \tan^3 \frac{\psi}{2}$ , where  $s$  and  $\psi$  are measured from the apse.

**Question (1915 STEP III Q507)**

Show how to evaluate the indeterminate forms  $\frac{0}{0}$  and  $\frac{\infty}{\infty}$ .

Find the limit when  $\theta = \frac{\pi}{2}$  of  $(\log(\sec \theta + \tan \theta)) \cdot (\log(\operatorname{cosec} \theta + \cot \theta))$ .

**Question (1916 STEP III Q509)**

Evaluate

$$\int x^2 e^x dx, \quad \int \frac{dx}{1+2x^2}, \quad \int_0^{\infty} x e^{-x^2} dx, \quad \int_0^{\frac{\pi}{2}} \sin^2 x dx.$$

**Question (1927 STEP III Q505)**

Evaluate  $\int \frac{x^2 dx}{1+x^4}$ , expressing the result in real form. Prove that  $\int_0^{\frac{\pi}{4}} \sqrt{\tan \theta} \cdot d\theta = \frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}} \log(\sqrt{2}-1)$ .

**Question (1916 STEP III Q509)**

Evaluate

$$\int_0^{\frac{1}{2}\pi} \cos^3 x dx, \quad \int_0^{\frac{1}{4}\pi} \frac{dx}{3+2\cos x}, \quad \int_0^1 \frac{x+2}{(4+x^2)(1-x)} dx.$$

**Question (1922 STEP II Q610)**

Integrate

$$\int \frac{1+(1+x)^{\frac{1}{2}}}{1-(1+x)^{\frac{1}{2}}} dx, \quad \int e^{ax} \cos bx dx, \quad \int \frac{x^2+4}{x^2+2x+3} dx.$$

Show that

$$\int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{1-x^2}} = \frac{1}{2} \log(2+\sqrt{3}).$$

**Question (1924 STEP II Q609)**

Find the values of  $\int \sec x dx$ ,  $\int x^n \log x dx$ ,  $\int \frac{dx}{x\sqrt{a^2+x^2}}$ . Show that

$$\int_b^a \frac{dx}{x\sqrt{(a-x)(x-b)}} = \frac{\pi}{\sqrt{ab}}, \quad (a > b > 0),$$

and that

$$\int_0^1 x^4 \sqrt{1-x^2} dx = \frac{5\pi}{256}.$$

**Question (1926 STEP II Q611)**

Integrate

$$\int \frac{dx}{1 + e^{2x}}, \quad \int \frac{d\theta}{\sin^2 \theta \cos^2(\theta + \alpha)},$$

and shew that if  $a$  and  $b$  are both positive and  $a > b$ ,

$$\int_0^\pi \frac{d\theta}{a + b \cos \theta} = \frac{\pi}{\sqrt{a^2 - b^2}}.$$

**Question (1927 STEP II Q611)**Evaluate  $\int \sec^3 x dx$ ,  $\int x^2 \sin^2 x dx$ ,  $\int \frac{(x-1)(x-4)}{(x-2)(x-3)} dx$ . Prove that

$$\int_0^{\frac{\pi}{2}} \cos^3 x \sin 5x = \frac{1}{2}.$$

**Question (1920 STEP III Q601)**

Evaluate the integrals

$$\int \sqrt{a^2 + x^2} dx, \quad \int \frac{dx}{(x-1)^{1/2}(x-2)}, \quad \int \left(\frac{1+x}{1-x}\right)^{1/2} dx.$$

**Question (1913 STEP I Q715)**

Evaluate:

$$(i) \int \frac{dx}{5 - 2x - 3x^2}, \quad (ii) \int \frac{3 \cos x + 4 \sin x}{2 \cos x - 3 \sin x} dx, \quad (iii) \int_0^1 \frac{dx}{1 + 2x \cos \alpha + x^2}.$$

**Question (1920 STEP I Q708)**

Prove that the intrinsic equation which represents the curve taken up by a uniform thin rod, when bent into a bow by means of a string attached to its ends, is

$$s = c \int_0^\psi \frac{d\psi}{\sqrt{\cos \psi - \cos \alpha}},$$

where  $c$  is a constant.  $s$ , the distance along the curve, is measured from the middle point of the rod.  $\psi$  is the angle between the tangent at any point and the tangent at the middle of the rod.  $\pm\alpha$  are the values of  $\psi$  at the ends of the rod. Find the law according to which the diameter of a thin rod of circular section must taper towards its ends in order that it may form a circle when bent so that the ends come together, and are held in that position by a flexible joint.

**Question (1921 STEP I Q708)**

The coordinates of any point of a surface are expressed in terms of two parameters  $u, v$ , the element of length on the surface being given by

$$ds^2 = Edu^2 + 2Fdu dv + Gdv^2.$$

Prove that the measure of curvature depends only on  $E, F, G$ , and their differential coefficients with respect to  $u$  and  $v$ . Prove that if  $F = 0$  for all values of  $u$  and  $v$ , and  $G$  is a function of  $v$  alone, the curves  $u = \text{constant}$  are geodesics.

**Question (1921 STEP I Q713)**

If  $\phi(z) \rightarrow 0$  uniformly as  $|z| \rightarrow \infty$ , prove that

$$\int_{\Gamma} e^{iz} \phi(z) dz \rightarrow 0$$

as  $R \rightarrow \infty$ , where  $\Gamma$  is the semicircle  $z = Re^{i\theta}, 0 \leq \theta < \pi$ . Hence, or otherwise, prove that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{1}{2}\pi.$$

**Question (1917 STEP II Q705)**

Prove that for a plane curve, with the usual notation,

$$\sin \phi = r \frac{d\theta}{ds}, \quad \cos \phi = \frac{dr}{ds}.$$

Prove that

$$\frac{d^2\phi}{ds^2} = \frac{\sin 2\phi}{r^2} - \frac{\cos \phi}{pr} - \frac{1}{\rho} \frac{dp}{ds},$$

where  $\rho$  is the radius of curvature at the point.

**Question (1918 STEP II Q708)**

State and prove Cauchy's theorem on the integral of an analytic function round a closed contour.

**Question (1918 STEP II Q710)**

Prove that if  $s_n$  is the sum of the first  $n$  terms of the Fourier series of a continuous and periodic function  $f(x)$ , then

$$\frac{s_1 + s_2 + \cdots + s_n}{n},$$

tends uniformly to  $f(x)$  when  $n \rightarrow \infty$ . State any extensions of this theorem with which you are familiar. What are the principal theorems concerning the convergence of Fourier series? How are they related to the theorem just stated?

**Question (1924 STEP II Q711)**

Evaluate

$$\int (1+x)\sqrt{1-x^2}dx, \quad \int_0^\pi \cos 2\theta \log(1+\tan \theta)d\theta, \quad \int_0^{\pi/2} \sin^2 \theta \cos 2\theta d\theta.$$

Prove that

$$(m+np) \int x^{m-1}(x^n+a^n)^p dx = x^m(x^n+a^n)^p + npa^n \int x^{m-1}(x^n+a^n)^{p-1} dx.$$

**Question (1924 STEP II Q712)**

Prove that the area of the loop of the curve  $y^2(a+x) = x^2(a-x)$  is  $2a^2(1 - \frac{\pi}{4})$  and that the volume formed by the revolution of the loop about the axis of  $x$  is  $2\pi a^3(\log 2 - \frac{2}{3})$ .

**Question (1922 STEP I Q814)**

Prove that the value of  $\int_{u_0}^{u_0+2\omega} \wp(u)du$  is independent of  $u_0$ , the integral being taken along any path not passing through a pole of  $\wp(u)$ . If  $l = \int_{u_0}^{u_0+2\omega} \wp(u)du$ ,  $l' = \int_{u_0}^{u_0+2\omega'} \wp(u)du$ , prove that

$$\omega l' - \omega' l = \pi i.$$

**Question (1923 STEP I Q807)**

Establish the formula for change of variable in a simple integral, stating carefully what conditions you assume. The transformation  $\sin x = y$  apparently gives

$$\int_0^\pi \cos^2 x dx = \int_0^0 \sqrt{1-y^2} dy = 0.$$

Explain this paradox.

**Question (1924 STEP I Q813)**

A function  $f(z)$  is regular (holomorphic) in the domain  $D$  obtained by excluding from the  $z$ -plane the two regions defined by

$$\begin{aligned}x < -\frac{1}{2} - \delta, \quad y \geq 0 \\x \geq +\frac{1}{2} + \delta, \quad y \leq 0\end{aligned}$$

( $z = x + iy$ ) ( $\delta$  being a positive constant), and satisfies throughout  $D$  the inequality

$$|f(z)| < Ke^{-2\pi|xy|},$$

where  $K$  is a constant. Shew, by integrating the function  $f(z)/\sin \pi z$  round a suitable contour, that

$$\int_L \frac{f(z + \frac{1}{2}) + f(z - \frac{1}{2})}{\cos \pi z} dz = 2if(0),$$

$L$  being any rectilinear path passing through the point  $z = 0$ , extending to infinity in both directions, and lying entirely in  $D$ ;  $L$  is to be described in the direction of increasing  $y$ . By taking  $f(z) = e^{i\pi z^2}$  and choosing  $L$  appropriately deduce the results

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}, \quad \int_{-\infty}^{\infty} \cos u^2 du = \sqrt{\frac{\pi}{2}}, \quad \int_{-\infty}^{\infty} \sin u^2 du = \sqrt{\frac{\pi}{2}}.$$

**Question (1919 STEP II Q813)**

Evaluate the integrals

$$\int \sec^4 \theta d\theta, \quad \int \tan^{-1} x dx, \quad \int \frac{dx}{(x+1)^2(x^2+1)}, \quad \int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}.$$

**Question (1914 STEP III Q805)**

Prove by contour integration or otherwise that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}, \quad \int_0^{\infty} \frac{e^{-x} \sin x}{x} dx = \frac{\pi}{4}.$$

**Question** (1922 STEP III Q809)

Find the differential equation which must be satisfied by magnetic potential in a magnetic material the intensity and direction of the magnetisation of which is known. A steel sphere is magnetised so that the direction and intensity of magnetisation is constant. It is surrounded by a close fitting shell of paramagnetic material of permeability  $\mu$ . The outer radius of the shell is  $n$  times the radius of the steel sphere. Show that the magnetic potential at external points is

$$\frac{9M\mu n^3}{\{2(\mu - 1)^2 - n^3(2\mu + 1)(\mu + 2)\}} \left( \frac{\cos \theta}{r^2} \right)$$

where  $M$  is the total magnetic moment of the steel sphere.