Question (1980 STEP II Q15)

The motion of particles in the solar system, under the influence of the sun's gravity, is described by the equations (in appropriate units)

$$r - r\dot\theta^2 = -1/r^2$$
$$r^2\dot\theta = h = \text{const.}$$

Using the second of these equations to give θ as a function of r, or otherwise, show that the first equation has the solution

$$\frac{1}{r} = \frac{1 + e\cos\theta}{h^2}$$

for any constant e. In the case $0 \le e < 1$, find the speed when the particle is nearest to the sun, and when it is furthest from it. A spaceship is in a circular orbit around the sun. Its velocity is increased instantaneously, parallel to itself, by a factor $5/4$. Show that it will reach out to a distance $25/7$ times its initial distance.

Question (1968 STEP III Q15)

The moment of momentum about a point O of a particle of mass m moving with velocity $\mathbf{u}$ is defined as the vector product $\mathbf{r} \times m\mathbf{u}$, where $\mathbf{r}$ is the vector drawn from O to the particle. Prove that, if O is such that $\mathbf{r}$ is parallel and the particle moves along a straight line with constant velocity, its moment of momentum about O is constant. A number of particles interact during a finite time interval. The mass of a typical particle is m_i, its velocity before the interaction is $\mathbf{u}_i$, and its velocity after the interaction is $\mathbf{v}_i$. We postulate that

$$\sum m_i\mathbf{u}_i = \sum m_i\mathbf{v}_i,$$

i.e. *the total momentum is conserved in the interaction* (postulate A). We postulate also that *there is a fixed point about which the total moment of momentum of all the particles is zero before and after the interaction* (postulate B). Show that A and B together imply that the *total moment of momentum about an arbitrary fixed point is conserved in the interaction* (principle C). Show also that C implies A.

Question (1966 STEP IIII Q8)

A particle is attached to the end of a light string which passes through a fixed ring. Initially the particle is moving in a horizontal circle, the string making an angle with the vertical. The string is then drawn upwards slowly through the ring until the distance of the particle from it has been halved. Assuming angular momentum is conserved, show that the string now makes an angle α' with the vertical, where

$$\frac{\sin^4\alpha'}{\cos\alpha'} = 8\frac{\sin^4\alpha}{\cos\alpha}.$$

Question (1977 STEP IIII Q13)

A heavy particle is projected horizontally with velocity V along the smooth inner surface of a sphere of radius a. Its initial depth below the centre is d and in the subsequent motion it never leaves the surface of the sphere. Show that, if u is the horizontal component of its velocity when the radius to the particle makes an angle θ with the downward vertical,

$$au \sin \theta = V(a^2 - d^2)^{\frac{1}{2}}.$$

Calculate the maximum and minimum heights attained by the particle and determine whether it moves upwards or downwards initially.

Question (1977 STEP IIII Q15)

A particle of unit mass orbits the sun under an inverse square law of gravity. Interplanetary gas imposes a resistive force which is $-k$ times the velocity, in magnitude and direction. Use the equation of motion in polar coordinates to show that the angular momentum decreases exponentially with time. If the resistive force is neglected show that the particle can move in a circular orbit, say with angular frequency ω. If $k \ll \omega$, so that k^2 can be neglected in comparison with ω^2, show that the radius of the orbit decreases by a fraction $4\pi k/\omega$ per revolution, and that the tangential velocity increases by a fraction $2\pi k/\omega$. Comment on the fact that as a result of the resistive force the velocity actually increases.

Question (1958 STEP IIII Q109)

The polar coordinates of a moving particle are (r, θ). Prove that the radial and transverse components of its acceleration are $\ddot{r} - r\dot{\theta}^2$ and $2\dot{r}\dot{\theta} + r\ddot{\theta}$. A particle moves under the action of a force directed towards the origin and of magnitude μ per unit mass (μ constant). Establish the equations of conservation of energy and moment of momentum: $\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu}{r} = E$, $r^2\dot{\theta} = h$, and prove that the differential equation of the orbit is $\frac{d^2u}{d\theta^2} + \left(u - \frac{\mu}{h^2}\right) = 0$, where $u = 1/r$. If the particle is initially at a point A at a distance c from the origin O, and its velocity is at right angles to OA and of magnitude V, find the conditions that the orbit shall be (i) an ellipse, (ii) an ellipse with its centre between O and A.

Question (1962 STEP III Q206)

Two particles P_1 and P_2 of masses m_1 and m_2 respectively are connected by a light inextensible string. P_1 lies on a smooth horizontal table, the string passes through a small hole O in the table, and P_2 hangs below the table. Initially P_1 is at distance a from O and moves at right angles to the radius OP_1 with speed V. In the subsequent motion the distance OP_1 at time t is r. Obtain an equation for this motion in the form $r^3 = f(r, a, V)$. Show that if at any subsequent instant P_1 again moves at right angles to OP_1, then at that instant r must equal a or $l + \sqrt{(l^2 + 2al)}$, where $l = m_1 V^2/m_2 g$.

Question (1957 STEP III Q406)

A particle P moves with acceleration λr^{-3} directed towards a fixed origin O, where r is the length of OP and λ is a positive constant. Using polar coordinates, when $r = d$, and $\theta = 0$ the direction of the velocity v is inclined at $\dfrac{\pi}{4}$ to the outward radius vector, and v^2 is positive and greater than unity and denoted by n^2. Establish the equation of motion in the form $a^2(\frac{du}{d\theta})^2 + u^2 = n^2 u$, where $u = 1/r$, and prove that the greatest value of r during the motion is given by $an(n^2 - 1)^{-\frac{1}{2}}$.

Question (1945 STEP II Q211)

Obtain the components of acceleration in polar coordinates and prove that, if a point moves under an acceleration to a fixed point, $r^2\dot{\theta} = h$, where h is a constant; prove also that, if $u = r^{-1}$,

$$\frac{d^2 u}{d\theta^2} + u = \frac{f}{h^2 u^2},$$

where f is the acceleration to the point. Deduce that, if a point moves in a circle under an acceleration towards a point on the circumference, f varies as r^{-5}. If $f = \mu r^{-5}$ and the point starts from a point at distance a from the centre of acceleration at right angles to the line joining it to the centre of acceleration, shew that the point will describe a semi-circular path, if the initial velocity is $\frac{1}{a^2}\sqrt{(\frac{\mu}{2})}$, and that it will reach the centre of acceleration in a time $\pi a^3/\sqrt{(8\mu)}$.

Question (1945 STEP III Q109)

A particle moves in a plane under a force directed towards a fixed point O and of magnitude $n^2 r$ per unit mass, where n is a constant and r is the distance of the particle from O. Initially the particle is at a point A at a distance a from O and has speed an in a direction making an angle $\pi/4$ with AO. Prove that the particle describes an ellipse, and find the lengths of the semi-axes.

Question (1945 STEP III Q308)

Find the formulae for the radial and transverse components of acceleration of a particle moving in a plane, the position of the particle at time t being described by the polar coordinates r, θ. A particle P of mass m moves in a plane under the action of a centre of force O (i.e. a force in the line OP), and of a force of magnitude $2mkv$ at right angles to the direction of motion, where v is the speed and k a constant. The particle is projected from O with velocity v_0. Prove that the angular velocity $\dot{\theta}$ remains constant throughout the motion. Find the path of the particle (i) when the central force is an attraction $3mk^2 r$ towards O, (ii) when the central force is a repulsion $mk^2 r$ away from O.

Question (1946 STEP III Q407)

A particle P moves under a central force of amount nk/r^{n+1} directed to a fixed point O, where $r = OP$, and k, n are positive constants with $n > 2$. Initially the particle is at great distance from O and is projected towards O with velocity v along a line that passes within a perpendicular distance p from O. Prove that in the subsequent motion

$$r^n \left(\frac{dr}{dt} \right)^2 = v^2 r^n - v^2 p^2 r^{n-2} + 2k.$$

Show that the particle will eventually again reach a great distance from O if

$$v^2 p^n (n-2)^{2n-1} > kn^{2n}.$$

Question (1930 STEP I Q101)

A right circular cone is circumscribed to a sphere. Shew that, if the radius of the sphere is given, the volume and the total surface area of the cone will each be a minimum when the height of the cone is twice the diameter of the sphere, and that the volume and surface of the cone are then twice those of the sphere.

Question (1925 STEP I Q111)

An electric motor which gives a uniform driving torque drives a pump for which the torque required varies with the angle during each revolution according to the law $T \propto \sin^2 \theta$: the mean speed of the pump is 600 rev. per min. and the mean horse-power required is 8. To limit the fluctuation of speed during each revolution, a flywheel is provided between the motor and the pump which successively stores and gives out energy. Shew that the energy thus successively stored and given out by the flywheel is approximately 70 foot pounds.

Question (1926 STEP I Q109)

The position of a point moving in two dimensions is given in polar co-ordinates r, θ: find the component velocities and acceleration along and perpendicular to the radius vector. The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu \theta^2$: find the polar equation of the path of the particle and also the component accelerations in terms of r and θ.

Question (1931 STEP I Q110)

A particle of mass m is describing an orbit in a plane under a force $\mu m r$ towards a fixed point at a distance r. Taking this point as origin of coordinates, shew that, if when the particle is at a point (a, b) it has a velocity with components u, v parallel to the axes, the orbit will be given by

$$\mu(bx - ay)^2 + (vx - uy)^2 = (av - bu)^2.$$

Question (1934 STEP I Q109)

Two equal particles are joined by a light inextensible string of length $\pi a/2$ and rest symmetrically on the surface of a smooth circular cylinder of radius a, the axis being horizontal. If the particles are slightly disturbed, show that one of the particles will leave the surface at a height

$$\frac{1}{5}(2\sqrt{2} - \sqrt{3})a$$

above the axis of the cylinder. The motion takes place in one plane.

Question (1930 STEP I Q109)

Two masses m, m' lie on a smooth horizontal table connected by a taut unstretched elastic string of modulus of elasticity λ and natural length l. The mass m is projected with velocity v in the direction away from the mass m'. Shew that the masses will collide after a time $\frac{\pi}{a}\frac{l}{v}$, where $a^2 = \frac{m+m'}{mm'}\frac{\lambda}{l}$.

Question (1925 STEP I Q207)

A particle moves under a force directed towards a fixed point O. Shew that its path lies in a plane and that pv is constant, where v is the velocity of the particle at any instant and p the length of the perpendicular from O to the tangent to the path. A particle is repelled from a centre of force O with a force μr per unit mass, where r is the distance of the particle from O. Shew that, if the particle is projected from a point P in any direction with velocity $OP\sqrt{\mu}$, its path is a rectangular hyperbola with O as centre.

Question (1929 STEP I Q210)

A light bar OA of length $2a$ with a particle of mass m attached to its middle point turns in a horizontal plane about a vertical axis through O; and a light bar AB, of the same length as OA and with a similar particle attached to its middle point, is freely jointed at A to the bar OA. A smooth guide compels the end B to move along a horizontal straight line Ox. The angle $AOx = \theta$. Shew that

$$\frac{d^2\theta}{dt^2} + \frac{4\omega^2 \sin\theta \cos\theta}{(5 - 4\cos^2\theta)^2} = 0,$$

where ω is the value of $\frac{d\theta}{dt}$ when $\theta = 0$.

Question (1942 STEP I Q205)

A bead moves without friction on a fixed circular wire; it is repelled from a fixed point of the wire by a force F which depends on the distance r between the bead and the fixed point. Find F in terms of r so that the reaction between the bead and the wire is the same for all positions of the bead.

Question (1913 STEP IIII Q209)

Prove that, when a particle describes a path under the action of a force directed to a fixed point, the radius vector drawn from the point to the particle describes equal areas in equal times. A particle of mass m is held on a smooth table. A string attached to this particle passes through a hole in the table and supports a particle of mass $3m$. Motion is started by the particle on the table being projected with velocity V at right angles to the string. If a is the original length of the string on the table, prove that when the hanging weight has descended a distance $a/2$ (assuming this possible) its velocity will be

$$\frac{\sqrt{3}}{2}\sqrt{(ga - V^2)}.$$

Question (1926 STEP IIII Q209)

An elastic string has one end fixed at A, passes through a small fixed ring at B and has a heavy particle attached at the other end. The unstretched length of the string is equal to $\frac{1}{2}AB$. The particle is projected from any point in any manner. Assuming that it will describe a plane curve, show that the curve is in general an ellipse.

Question (1933 STEP IIII Q210)

A particle of mass m moves in a plane, and is attracted towards a fixed origin O in the plane with a force mn^2r, where r denotes distance from O. It is projected from the point $(c, 0)$, the axes being rectangular, with velocity nb and in a direction inclined at an angle θ to the axis Ox. Shew that the path of the particle is the ellipse

$$b^2(x \sin\theta - y \cos\theta)^2 + c^2 y^2 = b^2 c^2 \sin^2\theta.$$

Shew further that the points of the plane which are accessible by projection from the given point with the given velocity lie within the ellipse

$$\frac{x^2}{b^2 + c^2} + \frac{y^2}{b^2} = 1.$$

Question (1936 STEP I Q307)

Two equal particles are connected by a light inelastic string of length $2l$. The particles are at rest on a smooth horizontal table at points A, B at a distance $l \sec\phi$ apart when the particle at B is caused to move on the table with velocity V in a direction making an acute angle 2ϕ with the direction of AB produced. Shew that the particle which was at B initially is again moving parallel to its initial direction of motion after time

$$\frac{l}{V}\{\sec\phi + (\pi + 2\phi)\operatorname{cosec}\phi\}.$$

Question (1914 STEP II Q310)

Prove that when a body describes a path round a centre of force the radius vector of the path sweeps out equal areas in equal times. Taking the semi-axis major of the earth's orbit round the sun as 93,000,000 miles and the eccentricity as $\frac{1}{60}$, find in miles per second the difference between the velocities at the two ends of the major axis.

Question (1922 STEP II Q303)

An aeroplane moving at a constant height above the ground describes a circle. Observations made at equal intervals of time t give angles α, β, γ for its altitude, while later it is noticed that the aeroplane passes directly over the point of observation. Shew that it completes a circle in time $\pi t / \cos^{-1}\left(\frac{\cot\gamma + \cot\alpha}{2\cot\beta}\right)$.

Question (1939 STEP III Q310)

A particle moves in a plane under an attraction $n^2 r$ per unit mass towards a fixed point O, where r denotes distance from O. The particle is projected from a point P, distant c from O, with velocity nb. Show that the path of the particle is an ellipse.

Let Q be a point of this ellipse at which the tangent is perpendicular to the tangent at P. Show that at Q the ellipse touches a second ellipse, which has O as centre, P as one focus, and major axis $2a$, where $a^2 = b^2 + c^2$. Hence determine what part of the plane is accessible by projection from P in different directions with the given velocity nb.

Question (1940 STEP IIII Q302)

A smooth hollow circular cylinder of radius R is fixed with its axis horizontal, and three smaller smooth uniform circular cylinders each of radius r and weight w are placed inside, two being in contact symmetrically placed about the lowest generator of the fixed cylinder, and the third resting on top of these two. Show that the position is one of equilibrium only if

$$R < r(1 + 2\sqrt{7}).$$

Question (1942 STEP IIII Q309)

A uniform disc of radius r and mass M is freely pivoted at a point on its circumference and hangs in a vertical plane. It is struck by a horizontal impulse F which acts in the plane of the disc, and is distant h below the pivot. Find the impulsive force at the pivot, and prove that if $h = r$ the energy imparted to the disc is $F^2/3M$.

Question (1931 STEP III Q403)

A number of small rings can slide freely on a smooth fixed circular wire, and each ring repels every other ring with a force which is measured by the product of their masses and the distance between them: show that, in equilibrium, the centre of gravity of the rings is the centre of the circle.

Question (1932 STEP III Q407)

The end P of a straight rod PQ describes with uniform angular velocity a circle of centre O, while the other end moves on a fixed line through O in the plane of the circle. The end Q' of an equal straight rod $P'Q'$ moves on the same fixed line through O. Prove that the velocities of Q, Q' are in the ratio $QO : OQ'$.

Question (1931 STEP IIII Q407)

Two strings, each of length l, are attached to a ceiling, and the lower ends are attached to a magnet of moment M, length l, and weight W. When the strings are vertical the magnet is in the magnetic meridian but with its north-seeking pole towards the south. Through what angle will it have to turn before it comes to another position of equilibrium? (Assume that the earth's magnetic field exists a couple $HM \sin \theta$ on the magnet when it makes an angle θ with the magnetic meridian.)

Question (1934 STEP IIII Q408)

A particle moves in a plane under a central force $\frac{\mu}{r^2}$ towards a point O. Prove that the orbit is a conic.

Find the equation of this conic if the particle is projected with velocity v from a point P at a distance a from O, if the initial direction of the particle makes an angle α with OP produced. Find also the velocity of the particle at each end of the major axis.

Question (1922 STEP I Q505)

A sphere of radius R rolls between two fixed horizontal straight lines which intersect at an angle 2α. Prove that, until the sphere slips through, its centre describes an ellipse of minor axis R and eccentricity equal to $\cos \alpha$.

Question (1913 STEP III Q609)

Define the hodograph and prove one of its properties. A particle describes a circle freely under the action of a constant force (not tending to the centre). Prove that the hodograph is the curve $r^2 = c^2 \sin \psi$ where r is a radius vector from the origin to a point on the curve and ψ is the angle the tangent at the point makes with r.

Question (1920 STEP I Q709)

A particle is acted on by a central force which varies inversely as the nth power of the distance. It is projected from a point at distance c from the centre in a direction making an angle β with the radius vector. The initial velocity is that which it would acquire after falling freely from rest at infinity to the point of projection. Show that the equation to the orbit is

$$\left(\frac{r}{c}\right)^{\frac{n-3}{2}} = \operatorname{cosec} \beta \sin \left(\beta - \frac{n-3}{2}\theta\right)$$

where θ is measured from the radius vector of the point of projection.

Question (1920 STEP I Q711)

Straight ripples move along the surface of a liquid of infinite depth under the influence of gravity and capillarity. Find the wave velocity. Show that corresponding to any given wave speed there are two possible wave lengths provided that the speed is greater than a certain fixed value; but that there is only one wave length corresponding to a given frequency.

Question (1921 STEP II Q703)

Prove that a circular orbit described under a central force varying as r^{-s} is stable if and only if $s < 3$. Obtain an expression for the apsidal angle and show that it can only equal π if $s = 2$. Find approximately the progressive motion of the apse when s differs from 2 by a small constant amount.

Question (1918 STEP III Q714)

Two spheres, radii a, b, have their centres at a distance c apart. Prove the approximate formula $p_{12} = 1/c$, showing that the error is of order $(a/c)^7$ or $(b/c)^7$.

Question (1925 STEP III Q702)

Show that a particle moving under the action of a fixed centre of gravitation describes a conic. Show that in an orbit of period T and small eccentricity e the polar angle θ is given as a function of the time by the equation

$$\theta = 2\pi t/T + 2e \sin 2\pi t/T.$$

Question (1925 STEP III Q708)

Show that the gravitational potential at a point P at a distance r from the centre of mass O of a gravitating system is approximately, if r is large compared with the dimensions of the system,

$$\gamma \left(\frac{M}{r} + \frac{A + B + C - 3I}{2r^3} \right),$$

where M is the mass of the system, A, B, C its principal moments of inertia at O, I its moment of inertia about the line OP, and γ the constant of gravitation. Show that the attraction of a distant particle of unit mass on a homogeneous spheroid of axes a, a, c produces a couple of magnitude

$$3\gamma M(a^2 - c^2)\sin\theta \cos\theta / 5r^3$$

about the diameter perpendicular to the plane containing OP and the polar axis.

Question (1922 STEP III Q803)

Find expressions for the components of acceleration along and perpendicular to the radius vector of a point whose polar coordinates are known as functions of time. Find the law of force under which a particle can describe the spiral $r = a\theta + b$, and find the velocity at any point of the path.