Question (1969 STEP I Q3)

Obtain the condition for the equation $ax^2 + 2bx + c = 0$ to have real roots, where a, b and c are real numbers. The real numbers p, q and r are such that none has unit modulus, and $p^2 + q^2 + r^2 + 2pqr = 1$. Prove that p, q, and r either all lie between $+1$ and -1, or all lie outside this range.

Question (1972 STEP I Q11)

Let $f(x) = ax^2+bx+c$ (a, b, c real, $a > 0$). Explain why the following statements are equivalent. (i) $f(x) \leq 0$ for some real number x. (ii) $b^2 - 4ac \geq 0$. The real numbers a_1, a_2, ..., a_n, b_1, b_2, ..., b_n are such that $b_1^2 - b_2^2 - ... - b_n^2 > 0$. By considering the expression $(b_1x - a_1)^2 - (b_2x - a_2)^2 - ... - (b_nx - a_n)^2$, or otherwise, prove that $(a_1^2 - a_2^2 - ... - a_n^2)(b_1^2 - b_2^2 - ... - b_n^2) \leq (a_1b_1 - a_2b_2 - ... - a_nb_n)^2$.

Question (1978 STEP I Q2)

Express $(a^2 + b^2 + c^2)(x^2 + \beta^2 + \gamma^2) - (a\alpha + b\beta + c\gamma)^2$ as the sum of three squares. Deduce that if α, β, γ are real numbers then

$$(a^4 + \beta^4 + \gamma^4)(a^2 + \beta^2 + \gamma^2) \geq (a^2 + \beta^2 + \gamma^2)^2.$$

Give necessary and sufficient conditions on α, β, γ for equality to hold.

Question (1971 STEP IIII Q3)

Let a, b, c be integers and let $f(x, y) = ax^2+2bxy+cy^2$. Show that there are integers p, q, r, s such that $ps - qr = 1$ and $f(x, y) = 2(px + qy)(rx + sy)$ if and only if a and c are even and $b^2 - ac = 1$.

Question (1959 STEP I Q202)

Prove that, if a, b, h are real numbers such that $a > 0$, $ab - h^2 > 0$, then

$$ax^2 + 2hx + b > 0$$

for all real values of x. If p, q, r are real, investigate the conditions under which

$$px^2 + 2qx + r > \rho$$

for all real values of x. Show that these conditions imply that $r > 2|q|$.

Question (1960 STEP I Q201)
If

$$x_1x_2+y_1y_2 = a_1, \quad x_2x_3+y_2y_3 = a_2, \quad x_3x_1+y_3y_1 = a_3, \quad x_1^2+y_1^2 = x_2^2+y_2^2 = x_3^2+y_3^2 = b,$$

prove that

$$b^3 - (a_1^2 + a_2^2 + a_3^2)b + 2a_1a_2a_3 = 0.$$

Deduce that, if the $2n$ equations

$$x_1x_2+y_1y_2 = a_1, \quad x_2x_3+y_2y_3 = a_2, \ldots, x_{n-1}x_n+y_{n-1}y_n = a_{n-1}, \quad x_nx_1+y_ny_1 = a_n,$$

$$x_1^2 + y_1^2 = x_2^2 + y_2^2 = \ldots = x_n^2 + y_n^2 = b,$$

for the $2n$ unknowns $x_1, \ldots, x_n, y_1, \ldots, y_n$, are consistent, there must be an algebraic relation connecting b and $a_1, a_2, \ldots, a_n$.

Question (1961 STEP I Q202)
Determine the limitations, if any, on the value of p if the expression

$$x^2(y^2 + 2y + 2) + 2x(y^2 + 2py + 2) + (y^2 + 2y + 2)$$

is greater than or equal to zero for all pairs of real values of x and y.

Question (1963 STEP I Q201)
Prove that the expression

$$5x^2 + 6y^2 + 7z^2 + 2yz + 4zx + 10xy$$

is positive for all real values of x, y, z, other than $x = 0$, $y = 0$, $z = 0$. Find a set of real values of x, y, z for which the expression

$$5x^2 + 6y^2 + 7z^2 - 2yz - 4zx - 10xy$$

is negative. (If you quote a general test in support of your arguments, you must prove it.)

Question (1961 STEP IIII Q303)
What conditions on the real numbers a, b, c are needed to ensure that

$$\frac{ax^2 + bx + c}{cx^2 + bx + a} = \lambda \qquad (1)$$

has a real root x for every real λ?

Question (1954 STEP I Q102)

Prove that, if $a > 0$ and $ac - b^2 > 0$, then $ax^2 + 2bx + c > 0$ for all real values of x. Examine whether it is possible to find real values of x, y and z which give a negative value to the expression

$$7x^2 + 10y^2 + z^2 - 6yz + zx - 8xy.$$

Question (1917 STEP II Q201)

If α, β denote the roots of a given quadratic equation $Ax^2 + Bx + C = 0$, find the quadratic of which the roots are $\frac{a\alpha^2+b\alpha+c}{a'\alpha^2+b'\alpha+c'}$ and $\frac{a\beta^2+b\beta+c}{a'\beta^2+b'\beta+c'}$. Prove that, if x be restricted to be real, $\frac{kx^2+kx+1}{x^2+kx+k}$ can have all values in case k is negative and not numerically less than $\frac{1}{4}$; that there are two values between which it cannot lie when k is negative and numerically less than $\frac{1}{4}$, or also when $k > 4$; and that there are two values between which it must lie in case k is positive and less than 4, these two values being coincident when $k = 1$.

Question (1918 STEP IIII Q205)

A train travels 525 miles; if its average rate had been $2\frac{1}{2}$ miles per hour faster, it would have taken 1 hour less; find its average rate.

Question (1916 STEP IIII Q503)

Find the conditions that the equation $ax^2 + 2bx + c = 0$ should have (i) both its roots positive and (ii) two equal roots.