Question (1944 STEP II Q410)

P is any point on an ellipse of which the foci are S and H. The distance SP is denoted by r and the angle HSP by θ. Show that the mean value of r with respect to arc is the semi-major axis a, and that the mean value of r with respect to θ is the semi-minor axis b. If Q is any point in the interior of the ellipse, show that the mean value of the distance SQ with respect to area is $a - \frac{b^2}{3a}$.

Question (1945 STEP II Q407)

Prove that the mean distance of points on the surface of a sphere of radius a from an external point distant c from the centre is $c + \frac{1}{3}\frac{a^2}{c}$. What is the value for an internal point?

Question (1947 STEP II Q409)

Prove that the mean value with respect to area over the surface of a sphere centre O and radius a of the reciprocal of the distance from a fixed point C is equal to the reciprocal of OC if C is outside the sphere, but equal to the reciprocal of the radius a if C is inside the sphere.

Question (1947 STEP II Q305)

Prove that, if $f(x)$ is a function of x which has a derivative $f'(x)$ for all values of x between a and b inclusive, and if $f(a) = f(b)$, there is at least one value ξ between a and b for which $f'(\xi) = 0$.
Deduce from this theorem that, for some ξ between a and b,

$$\text{(i)} \quad \frac{\phi(b) - \phi(a)}{\psi(b) - \psi(a)} = \frac{\phi'(\xi)}{\psi'(\xi)},$$

and that, for another ξ,

$$\text{(ii)} \quad \frac{\phi(\xi) - \phi(a)}{\psi(b) - \psi(\xi)} = \frac{\phi'(\xi)}{\psi'(\xi)},$$

where in each case it is assumed that $\phi'(x), \psi'(x)$ exist for all values of x between a and b inclusive, and that $\psi'(x)$ does not vanish for any x between a and b.

Question (1944 STEP III Q206)

The position of a point moving in two dimensions is given by polar coordinates r, θ; find the component velocities and accelerations along and perpendicular to the radius vector. The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$; find the polar equation of the path and the component accelerations in terms of r and θ.

Question (1927 STEP I Q106)

Find the mean value of the distance of a point on the circumference of a circle of radius a from $2n$ points arranged at equal distances along the circumference. Shew that when $n \to \infty$ the mean is $\dfrac{4a}{\pi}$.

Question (1929 STEP I Q113)

AB and CD are perpendicular diameters of a circle. Find the mean value of the distance of A from points on the semicircle CBD and also the mean value of the reciprocal of that distance. Shew that the product of these means is

$$\frac{8\sqrt{2}\log_e(1+\sqrt{2})}{\pi^2}.$$

Question (1937 STEP II Q207)

State (without proof) Rolle's theorem, and deduce that there is a number ξ between a and b such that

$$f(b) - f(a) = (b-a)f'(\xi), \quad (1)$$

explaining what conditions must be satisfied by the function $f(x)$ in order that the theorem may be valid. If $f(x) = \sin x$, find all the values of ξ between a and b which satisfy the equation (1) when $a = 0$ and $b = 3\pi/2$. Illustrate the result with reference to the graph of $\sin x$.

Question (1925 STEP IIII Q206)

Define the mean value of $f(x)$ with respect to x for values of x lying in an interval (a, b). A point moves along a straight line in such a way that

$$v_t = v_s + ks,$$

where v_t, v_s are the mean values of the velocity with respect to the distance travelled s and the time taken t respectively, and k is a constant. Shew that s, t satisfy the equation

$$\frac{ds}{s} = \frac{dt}{t}\{1 + kt \pm \sqrt{kt(2+kt)}\}.$$

Interpret this solution in the case $k = 0$, and shew on general grounds that a negative value of k is inadmissible.

Question (1924 STEP II Q308)

What is meant by the Mean Value of a function $f(x)$ with respect to a variable x? A point moves from rest along a straight line in such a way that its average velocity with respect to distance travelled bears a constant ratio k to that with respect to time elapsed. Shew that $k > 1$.

Question (1939 STEP III Q303)

The functions $\phi(x)$ and $\psi(x)$ are differentiable in the interval $a < x < b$; and $\psi'(x) > 0$ for $a < x < b$. Prove that there is at least one number ξ between a and b such that

$$\frac{\phi(\xi) - \phi(a)}{\psi(b) - \psi(\xi)} = \frac{\phi'(\xi)}{\psi'(\xi)}.$$

If $\phi(x) = x^2$ and $\psi(x) = x$, find a value of ξ in terms of a and b.

Question (1938 STEP III Q410)

If r denotes distance from a focus of an ellipse, find the mean value of r with respect to angular distance from the major axis for points on the perimeter of the ellipse. Determine also for the ellipse the mean value of r with respect to area, stating the result in terms of the eccentricity e and the semi-latus rectum λ.