

Question (1971 STEP I Q11)

Sketch the graph of the function

$$\phi_n(x) = e^{-x} \left(1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} \right) - k,$$

where k is a constant, $0 < k < 1$; distinguish as you think fit between different values of n . Show that there is just one positive value of x for which

$$1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} = ke^x.$$

Denoting this by x_n , show that $x_n < x_{n+1}$. [It may be assumed that, for any m , $x^m e^{-x} \rightarrow 0$ as $x \rightarrow \infty$.]

Question (1981 STEP I Q16) (i) Show that $(1+t)(1-t+t^2+\dots+(-1)^n t^n) = 1 + (-1)^n t^{n+1}$.

(ii) Using this result for $n = 1$ show that

$$t - \frac{1}{2}t^2 < \ln(1+t) < t \quad \text{for } t > 0.$$

(iii) Prove that

$$\left| \ln(1+t) - \sum_{r=1}^n (-1)^{r+1} \frac{t^r}{r} \right| < \frac{t^{n+1}}{n+1} \quad \text{for } t > 0.$$

Question (1969 STEP II Q10)

Write down the expansions of e^x and $(1-x)^{-1}$ as power series in x . Show that, for $0 < a < \frac{1}{2}$,

$$\int_0^a \frac{e^x - 1}{x} dx < a + \frac{1}{4}a^2(1 - \frac{1}{8}a)^{-1}.$$

Show also that

$$1.80 < \int_0^1 \frac{e^x - 1}{x} dx < 1.83.$$

Question (1976 STEP II Q3)

Prove that if $|x| \leq \frac{1}{2}$ then $x \geq \log(1+x) \geq x - x^2$. By taking logarithms, or otherwise, show that for any positive integer k

$$\left(1 - \frac{1}{n^2}\right) \left(1 - \frac{2}{n^2}\right) \dots \left(1 - \frac{kn}{n^2}\right) \rightarrow e^{-k^2/2}$$

as $n \rightarrow \infty$.

Question (1980 STEP II Q4)

By considering the derivative of $x - \sin x$ show that $x \geq \sin x$ for all $x \geq 0$. By considering the repeated derivatives of $\sin x - x + x^3/3!$ show that $\sin x \geq x - x^3/3!$ for all $x \geq 0$. More generally, show that

$$\sum_{r=0}^{2m} (-1)^r \frac{x^{2r+1}}{(2r+1)!} \geq \sin x \geq \sum_{r=0}^{2m-1} (-1)^r \frac{x^{2r+1}}{(2r+1)!}$$

for all $x \geq 0$ and $m \geq 1$. Deduce that

$$\left| \sum_{r=0}^{n-1} (-1)^r \frac{x^{2r+1}}{(2r+1)!} - \sin x \right| \leq \frac{x^{2n+1}}{(2n+1)!}$$

for all $x \geq 0$. [The power series expansion of $\sin x$ must not be used.]

Question (1982 STEP III Q7)

If

$$y = \sin^{-1} x$$

show that

$$(1 - x^2)y'' = xy',$$

and hence using Leibniz' Theorem evaluate $y^{(n)}(0)$. Write down the MacLaurin series for $\sin^{-1} x$. By considering the series expansions of the two functions term by term, show that

$$\sin^{-1} x < \frac{x}{1-x^2} \quad \text{for } 0 < x < 1.$$

Question (1960 STEP II Q406)

Obtain a series expansion of $\log_e \{1 + (1/x)\}$ in ascending powers of $1/(2x+1)$. For what ranges of values of x is this expansion valid? Prove that if x is strictly positive, for what

$$\frac{2x+1}{2x(x+1)} > \log_e \left(1 + \frac{1}{x}\right) > \frac{2}{2x+1}.$$

Question (1950 STEP III Q310)

Using the equation

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2}$$

show that, if $x > 0$, $\tan^{-1} x$ lies between $x - \frac{x^3}{3}$ and $x - \frac{x^3}{3} + \frac{x^5}{5}$. Use this result to evaluate $\tan^{-1} \frac{1}{11}$ correct to five places of decimals.

Question (1955 STEP III Q308)

Show that

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}.$$

By using the series expansion of $\arctan x$, or otherwise, evaluate π with an error of less than 10^{-3} .

Question (1916 STEP I Q111)

Show that

$$\frac{2n + (n + 1)x}{2n + (n - 1)x} < \sqrt[n]{1 + x},$$

if $x > 0$ and $n > 1$. Show also that the difference between the two functions, when x is small, is approximately

$$\frac{n^2 - 1}{12n^3} x^3.$$

Question (1925 STEP I Q109)

Prove that if $f(x)$ and its first two derivatives are continuous in $0 \leq x \leq a$ ($a > 0$), and $x, x + h$ are any two points of this interval, then

$$f(x + h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x + \theta h),$$

where θ is some number satisfying $0 < \theta < 1$. By taking $h = -x$, or otherwise, prove that, if $f(0) = 0$, and $f''(x) > 0$ in $0 < x < a$, then $f(x)/x$ is an increasing function of x in this interval. Explain the geometrical significance of this result. Deduce that $(\sin x)/x$ decreases as x increases from 0 to π .

Question (1921 STEP I Q105)

A function $f(x)$ and as many of its derivatives as are required are single valued and continuous for values of x in the neighbourhood of a given value a . If $\psi(x) = f(a + h) - f(a + h - x) - xf'(a + h - x) - \dots - \frac{x^n}{n!} f^{(n)}(a + h - x)$, show that $f(a + h) = f(a) + hf'(a) + \dots + \frac{h^n}{n!} f^{(n)}(a) + \psi(h)$, so that $\psi(h)$ is the remainder in Taylor's theorem. Prove that $\psi'(x) = \frac{x^n}{n!} f^{(n+1)}(a + h - x)$, and deduce $\psi(h) = \frac{1}{n!} \int_0^h x^n f^{(n+1)}(a + h - x) dx$. Obtain this in the form

$$\frac{h^{n+1}}{n!} \int_0^1 f^{(n+1)}(a + th)(1 - t)^n dt,$$

and discuss its behaviour as $n \rightarrow \infty$, when

- (i) $a = 1, f(x) = x^m (-1 < x < 1)$;
- (ii) $a = 0, f(x) = e^x$.

Question (1916 STEP I Q104)

Assuming the logarithmic series, obtain superior and inferior limits for the remainder after n terms in the expansions in ascending powers of x of (i) $\log_e(1+x)$, (ii) $\log_e\{1/(1-x)\}$, (iii) $\log_e\{(1+x)/(1-x)\}$. Prove that if these series are used to calculate $\log_e(128/125)$ correct to ten places of decimals, six terms must be taken in each of the first two series, while two are sufficient in the third case; and, using the tables provided, obtain in each case the remainders correct to two significant figures.

Question (1940 STEP II Q207)

Prove that, if $f(x)$ is a function whose differential coefficient $f'(x)$ is positive throughout a given interval, then $f(x_2) > f(x_1)$, if $x_2 > x_1$, where x_1, x_2 are any two values of x in the interval.

Prove that

$$x - \frac{x^3}{6} + \frac{x^5}{120} > \sin x$$

for all positive values of x , and that

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \sin x > \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) \cos x$$

when $0 < x < \pi$.

Question (1919 STEP III Q309)

Prove that, if $\cos \beta = \cos \theta \cos \phi + \sin \theta \sin \phi \cos \alpha$, and $\sin \alpha = e \sin \beta$

$$d\theta\{1 - e^2 \sin^2 \phi\}^{\frac{1}{2}} + d\phi\{1 - e^2 \sin^2 \theta\}^{\frac{1}{2}} = 0.$$

Question (1940 STEP III Q305)

The function $f(x)$ has a continuous second derivative $f''(x)$ in the interval $[a, b]$; prove that, if $a < x < b$,

$$\frac{f(x) - f(a)}{x - a} - \frac{f(b) - f(a)}{b - a} = \frac{1}{2}(b - x)f''(\lambda),$$

where λ is a number between a and b .

The value of a certain function $f(x)$ is known for $x = a$ and for $x = b$, and the value at an intermediate point is found by the "rule of proportional parts". Prove that the error in the value so found is not greater than $\frac{1}{8}(b - a)^2 M$, where M is the upper bound of $f''(x)$ in $[a, b]$.

If the value of $\sin x$ is known accurately at intervals of one-tenth of a degree, prove that the value for any x can be found by the rule of proportional parts with an error which is less than 4×10^{-7} .

Question (1917 STEP I Q409)

Prove that, under certain conditions

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x+\theta h), \quad 0 < \theta < 1.$$

Give examples of cases in which the theorem does not hold. Expand y in terms of x by Maclaurin's Theorem, knowing that $(1-x^2)y'' - xy' - y = 0$ and that, when $x = 0$, $y = 1$ and $y' = 1$.

Question (1913 STEP II Q608)

State Maclaurin's Theorem for the expansion of $f(x)$. Apply this method to the expansion of $\sin\left(x + \frac{\pi}{4}\right)$ in ascending powers of x .