Question (1972 STEP I Q6)

Let C_1 be the plane curve whose polar equation is $r\theta = 1$, $\theta \geq \pi$ and let C_2 be the curve whose equation is $r(\theta + \pi) = 1$, $\theta \geq \pi$. Show that these curves do not cross. Find the spiral-shaped area enclosed by C_1, C_2 and the line segment $\theta = \pi$, $1/2\pi \leq r \leq 1/\pi$. Find the area of the snail-shaped region bounded by the arc $\pi \leq \theta \leq 3\pi$ of C_1 and the line segment $\theta = \pi$, $1/3\pi \leq r \leq 1/\pi$.

Question (1972 STEP I Q7)

(i) Explain why the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ) given by $r = (x^2 + y^2)^{\frac{1}{2}}$, $\theta = \tan^{-1}(y/x)$ needs further comment. (ii) Let a, b, c and d be real numbers with $ad - bc \neq 0$, and let $T(x) = \frac{ax+b}{cx+d}$. Assuming that $T(x) \neq x$, find a necessary and sufficient condition in terms of a, b, c and d for the identity $T(T(x)) \equiv x$ to hold. Show that if T satisfies this condition and $c \neq 0$ then there are two distinct solutions (possibly complex) of $T(x) = x$.

Question (1973 STEP I Q15)

Sketch the plane curve C whose polar equation is $r = a\text{cosec}^2\frac{1}{2}\theta$, where $0 < \theta < 2\pi$. Calculate: (i) the length of the arc C_1 consisting of those points of C such that $\frac{1}{2}\pi \leq \theta \leq \pi$; (ii) the area enclosed by the arc C_1 and the radii $\theta = \frac{1}{2}\pi$ and $\theta = \pi$.

Question (1975 STEP I Q14)

Sketch on the same diagram the curves given in polar co-ordinates (r, θ) by the equations $r = \frac{1}{2}a(1 + \cos\theta)$ and $r = a\theta$ $(a > 0,\ 0 \leq \theta \leq 2\pi)$. Find the area of the region consisting of all those points (r, θ) such that $\frac{1}{3}\pi \leq \theta \leq 2\pi$ and $\frac{1}{2}a(1 + \cos\theta) \leq r \leq a\theta$.

Question (1978 STEP I Q11)

Sketch and describe the three curves given in polar coordinates by

$$(i)\ r = \sin\theta \quad (0 < \theta < \pi);$$
$$(ii)\ r^{-1} = \sin\theta \quad (0 < \theta < \pi);$$
$$(iii)\ r^{-2} = \sin 2\theta \quad (0 < \theta < \pi/2).$$

[No credit will be given for solutions obtained by numerical methods alone.]

Question (1970 STEP II Q11)

A closed curve is given in polar coordinates by the equation

$$r = a(1 - \cos\theta).$$

Show that the tangent at the point θ is inclined at an angle $\psi = \frac{1}{2}\theta$ to the axis $\theta = 0$. Find the radius of curvature at the point θ.

Question (1971 STEP II Q2)

Sketch the curve whose equation, in polar coordinates, is

$$\frac{l}{r} = 1 + e\cos\theta,$$

e being a positive constant; distinguish between the cases $e < 1$, $e = 1$, $e > 1$. By using the substitution

$$\cos\phi = \frac{\cos\theta + e}{1 + e\cos\theta} \quad (0 \leq \theta \leq \pi),$$

or otherwise, find the area enclosed by the curve when it is closed.

Question (1974 STEP II Q3)

Sketch the '$2m$-rose' defined in polar coordinates by $r = |\sin m\theta|$, for $m = 1, 2, 3$. Show that for all integers $m > 0$ the total area of the petals is independent of m, and evaluate this area.

Question (1977 STEP II Q1)

A curve is given parametrically in plane polar coordinates by $(r, \theta) = (e^t, 2\pi t)$ $(0 \leq t < \infty)$. Sketch the section of the curve for $n \leq t \leq n+1$, where n is an integer. Calculate the length of this section, and the area enclosed by it and the line $\theta = 0$, $r^n \leq r \leq r^{n+1}$.

Question (1982 STEP II Q10)

Sketch the curve whose equation in polar coordinates is

$$r = 1 - \frac{5}{6}\sin\theta.$$

Find the range of real values of b for which the simultaneous equations

$$(x^2 + y^2 + \frac{5}{6}y)^2 = x^2 + y^2$$

$$y = b$$

have a real solution.

Question (1983 STEP II Q8)

Sketch the curve given by the equations

$$x = a(\theta + \sin\theta)$$
$$y = a(1 - \sin\theta), \quad a > 0.$$

Find the area under the curve between two successive points where $y = 0$.

Question (1984 STEP II Q2)

A curve is given in polar coordinates by $r(\theta)$ for $0 \leq \theta \leq \pi$, and it is rotated about the axis $\theta = 0$ to form a solid of revolution. Derive a formula for the surface area of the solid. Calculate the area of the surface so formed in the case $r(\theta) = ae^{k\theta}$.

Question (1981 STEP III Q6)

At time $t = 0$, 4 insects A, B, C and D stand at the corners of a square of side a. For time $t > 0$ each insect crawls with constant speed v in the direction of the next insect in cyclic order (that is, A crawls towards B, B towards C, and so on). Show that the insects meet after a time a/v and that they encircle the centre of the square an infinite number of times. [Hint: Use polar coordinates.]

Question (1981 STEP III Q7)

A circle of radius a rolls without slipping around the outside of a circle of radius $2a$. Show that the arc length of a curve traced out by a point on its circumference is $24a$.

Question (1974 STEP IIII Q13)

A mouse runs along a straight line $y = 0$ with uniform speed V_1. When the mouse is at the point $x = 0$, $y = 0$ it is spotted by a cat at the point $x = 0$, $y = b$ which immediately gives pursuit. The cat runs with constant speed V_2 $(> V_1)$ and is always directed at the fleeing mouse. Make a qualitative sketch of the path of the cat (i) relative to a fixed frame of reference, and (ii) relative to a frame of reference moving with the mouse. Let (r, θ) be polar coordinates in this latter frame of reference, the mouse being at $r = 0$ and θ being measured from its direction of motion. Show that the differential equation of the path of the cat is

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{V_2}{V_1}\csc\theta - \cot\theta.$$

Integrate this, and show that if $V_2 = 2V_1$ the cat catches the mouse after a time of pursuit $2b/3V_1$.

Question (1975 STEP IIII Q5)

A solid cone is described by the following equations (in cylindrical polar coordinates (r, ϕ, z)):

$$r \leq -z \tan\alpha \quad (\alpha < \pi/2) \qquad (1)$$
$$z \leq 0 \qquad (2)$$
$$z \geq mr\sin\phi - a \quad (m < \cot\alpha) \qquad (3)$$

α, m and a are constants with $a > 0$. Sketch the cone. The cone is placed on its side on a plane, with its vertex at a point O. The cone is in contact with the plane along the line segment OP, which is initially of length $a\sec\alpha$. The cone now rolls on the plane, the vertex remaining at O. Obtain the polar equation of the locus of the point P in terms of the distance R from O to P, and the angle between the line OP and its initial direction. What conditions are required on α to ensure that the curve is closed?

Question (1961 STEP I Q309)

A string of length π is attached to the point $(-1, 0)$ of the circle $x^2 + y^2 = 1$, and is wrapped round the circle so that its other end is at the point $(1, 0)$. The string is unwound, being kept taut, and is wound up again the other way; if S is the path of the end of the string, show that S has length $2\pi^2$, and find the area enclosed by S.

Question (1962 STEP I Q309)

A man is unwinding a string wrapped round a smooth closed convex curve $ABCD$ on a piece of paper. When AB (of length a) is unwound, that part of the string becomes impregnated with ink. Prove that when ABC has been unwound, the area of the ink blot is $\pi\int_B^C s\,d\psi$, where s is the arc-length measured from the mid-point of AB, and ψ is the angle between the tangent and some fixed direction.

Question (1959 STEP II Q108)

Sketch the curve C whose equation in polar coordinates is

$$r^2 = a^2\cos 2\theta,$$

where $a > 0$ and it is understood that r may take negative as well as positive values. Show that the perimeter s of C is given by

$$s = 4a\int_0^{\pi/4} \frac{d\theta}{\sqrt{\cos 2\theta}}.$$

By means of the substitution $t = \tan^4\theta$, or otherwise, express s in terms of the function $B(p, q)$ defined by

$$B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt \quad (p > 0, q > 0).$$

Question (1962 STEP II Q110)

Sketch the curve $r = a(1+\cos\theta)$ and find its total length. Find also the perpendicular distance between the origin and the tangent to the curve at the point (r, θ).

Question (1963 STEP II Q108)

Calculate the volume of the solid of revolution formed by rotating the cardioid $r = a(1 - \cos\theta)$ about the line $\theta = 0$.

Question (1954 STEP IIII Q201)

$P_1, P_2, \ldots, P_N$ are N points lying on a straight line l. For $n = 1, 2, \ldots, N$, the polar coordinates measured from P_n as vertex and l as axis are r_n, θ_n, and $a_1, a_2, \ldots, a_N$ are constants. Show that every curve of the family

$$\sum_{n=1}^{N} \frac{a_n}{r_n} = \text{constant}$$

cuts every curve of the family

$$\sum_{n=1}^{N} a_n \cos\theta_n = \text{constant}$$

orthogonally.

Question (1955 STEP IIII Q210)

Let (r, θ) denote polar coordinates in the plane. (i) Find the area lying within both the circle $r = 1$ and the cardioid $r = 1 + \sin\theta$. (ii) Find the area lying within both the circle $r = \sqrt{2}\sin\theta$ and the lemniscate $r^2 = \cos 2\theta$.

Question (1956 STEP IIII Q210)

Give a rough sketch of the curve

$$y^2 - 2x^3y + x^7 = 0,$$

and find the area of the loop.

Question (1957 STEP IIII Q208)

Sketch the curve

$$y^2(1 + x^2) = (1 - x^2)^2,$$

and find the area of its loop.

Question (1951 STEP IIII Q310)

Sketch the curve

$$r(1-2\cos\theta) = 3a\cos 2\theta,$$

and find the equations of its asymptotes.

Question (1953 STEP IIII Q307)

Find the area and centroid (centre of mass) of the plane region whose boundary is given in polar co-ordinates by $r = a(1+\cos\theta)$.

Question (1956 STEP IIII Q308)

Sketch the curve whose equation, in Cartesian coordinates, is

$$x^4 - 2xy^2 + y^4 = 0.$$

Question (1951 STEP II Q107)

Sketch the curve whose equation in polar coordinates is $r = 1 + \cos 2\theta$. Prove that the length of the curve corresponding to $0 \leq \theta \leq 2\pi$ is

$$8 + \frac{4}{\sqrt{3}}\log(2+\sqrt{3}).$$

Question (1953 STEP II Q109)

Sketch the curve whose equation in polar coordinates is

$$r = \sin 3\theta - 2\sin\theta.$$

Find any maximum or minimum values of r. Prove that each of the smaller loops of the curve has area less than 0.005.

Question (1955 STEP II Q104)

Light emitted from the point A on the circumference of a circle of centre O and radius a is reflected at the circumference in the plane of the circle. Prove that the once reflected rays all touch the curve

$$x = \tfrac{1}{4}a(2\cos t - \cos 2t),$$
$$y = \tfrac{1}{4}a(2\sin t - \sin 2t),$$

where axes have been taken at O such that A is the point $(-a, 0)$. Identify this curve as a cardioid, and show its position in a sketch.

Question (1957 STEP II Q104)

If ϵ is small in magnitude compared with unity, show that the perimeter of the curve

$$r = 1 + \epsilon \cos\theta$$

is approximately $\frac{1}{2}\pi(4+\epsilon^2)$.

Question (1957 STEP II Q106)

Sketch the curve

$$(x+y)(x^2+y^2) = 2xy,$$

and obtain the area of its loop.

Let $X = x+y$ and $Y = x-y$, notice that

$4xy = X^2 - Y^2$

so our equation becomes

$$\begin{aligned} \frac{1}{2}(X^2-Y^2) &= X(X^2 - \frac{1}{2}(X^2-Y^2)) \\ &= \frac{1}{2}X(X^2+Y^2) \\ \Rightarrow \qquad Y^2(X+1) &= X^2 - X^3 \\ \Rightarrow \qquad Y^2 &= \frac{X^2-X^3}{X+1} \end{aligned}$$

Question (1952 STEP II Q410)

Obtain an expression for the area of a closed oval curve of polar equation $r = r(\theta)$ in the two cases when the pole ($r = 0$) is inside the curve and when it is outside the curve. A circle of radius a has centre C, and a point O is taken at distance $c(< a)$ from its centre. The foot of the perpendicular from O to a tangent to the circle is P. Show that the locus of P is a closed curve of area $\pi(a^2 + \frac{1}{2}c^2)$.

None

Question (1953 STEP II Q410)

Derive the polar equation of a plane curve whose tangent is inclined at a constant angle α to the radius vector from O. Prove that the length d of a chord subtending an angle β at O is given by

$$d = r(1 - 2e^{\beta\cot\alpha}\cos\beta + e^{2\beta\cot\alpha})^{\frac{1}{2}},$$

where r is the radius vector to the end of the chord nearer to O. Prove also that the line joining a point of the curve to its centre of curvature subtends a right angle at O.

Question (1954 STEP II Q409)

Find the length of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

The part of the curve lying between two consecutive cusps is rotated about the line joining the cusps to form a solid of revolution. Prove that the area of its surface is $3\pi a^2/5\sqrt{2}$.

Question (1954 STEP II Q410)

Establish that the radius of curvature of a plane curve whose pedal equation is $r = r(p)$ is rdr/dp. Show that the square of the length of the tangent from the pole to the circle of curvature at a general point is given by $\frac{d}{dq}(r^2q)$, where $q = 1/p$. Hence show that if a curve is such that all its circles of curvature pass through a certain fixed point, then it must itself be a circle.

Question (1955 STEP II Q408)

Sketch the curve given by the plane polar equation $r^3 = a^3(1 + 2\cos\theta)$. Prove that the area enclosed by the curve is $a^2(\frac{2}{3}\pi + \sqrt{3})$, and the volume enclosed by the surface obtained by revolving the curve about the line $\theta = 0$ is $6\pi a^3\sqrt{3}/5$.

Question (1957 STEP II Q409)

Discuss the general nature of the plane curve whose polar equation is $r = \dfrac{a}{\theta^2 - 1}$ for values of $\theta > 1$. Prove that one of the bisectors of the angle between the radius vector and the normal is inclined at an angle $\tan^{-1}\theta$ to the radius vector, and find an expression for the length of arc from the origin to the given point.

Question (1946 STEP IIII Q308)

A point P varies so that $PA.PA' = a^2$, where A and A' are fixed points with midpoint O and $AA' = 2a$. Sketch the curve and find its equation in polar co-ordinates, where $OP = r$. Determine the relation between r and p, the length of the perpendicular from O to the tangent at P. Hence, or otherwise, express the radius of curvature as a function of r.

Question (1947 STEP II Q108)

Trace the curve $(x^2 + y^2)^2 = 16axy^2$, and find the areas of its loops.
Prove that the smallest circle that will completely circumscribe the curve has radius $3\sqrt{3}a$.

Question (1944 STEP II Q407)

A plane curve is such that the tangent at any point P is inclined at an angle $(k+1)\theta$ to a fixed line Ox, where k is a positive constant and θ is the angle xOP. The greatest length of OP is a. Find a polar equation for the curve. Sketch the curves for the cases $k = 2, k = \frac{1}{2}$.

Question (1945 STEP II Q408)

Sketch the curve whose polar equation is $r^2 = a^2(1 + 3\cos\theta)$ and find the area it encloses.

Question (1946 STEP II Q410)

A curve whose polar equation is $f(r, \theta) = 0$ has pedal equation $F(r, p) = 0$. Prove that the curve whose polar equation is $f(r^n, n\theta) = 0$ has pedal equation $F(r^n, pr^{n-1}) = 0$.

Question (1947 STEP II Q406)

Trace the curve $16a^3y^2 = b^2x^2(a - 2x)$, where a and b are positive, and find the area enclosed by the loop.
If $16a^2 = 3b^2$, show that the perimeter of the loop is $\frac{1}{2}b$.

Question (1947 STEP II Q204)

P is a point on a bar AB which moves in a plane and returns to its original position after completing exactly one revolution. S_P, S_A, S_B are the areas of the closed curves described by P, A, B respectively. Prove that

$$S_P = \frac{aS_B + bS_A}{a + b} - \pi ab,$$

where $a = AP, b = PB$.
[The areas are reckoned positive if the bounding curve is described anti-clockwise, otherwise they are reckoned negative.]

Question (1923 STEP I Q114)

If A is the area bounded by the curve $r = f(\theta)$ and the straight lines $\theta = \theta_1$, $\theta = \theta_2$, show that the cartesian coordinates of the centroid of the area are ξ, η, where

$$A\xi = \frac{1}{3}\int_{\theta_1}^{\theta_2} r^3 \cos\theta d\theta, \quad A\eta = \frac{1}{3}\int_{\theta_1}^{\theta_2} r^3 \sin\theta d\theta.$$

Question (1924 STEP I Q113)

Find the area of a loop of the curve

$$r = 3\sin 2\theta + 4\cos 2\theta.$$

Question (1927 STEP I Q114)

Find the area of a loop of the curve

$$r^2 = a^2(\sin 2\theta + 2\sin\theta).$$

Question (1936 STEP I Q108)

Prove that the evolute of the logarithmic spiral $r = ae^{\alpha\theta}$ is an equal spiral.

Question (1937 STEP I Q110)

By transforming to polar coordinates, or otherwise, find the area of the loop of the curve

$$x^3 + y^3 = 3axy.$$

Question (1939 STEP I Q109)

The perpendicular from the origin O on to the tangent at a point P of a plane curve C is of length p and is inclined to the axis of x at an angle α. Show that the coordinates of P are

$$(p\cos\alpha - p'\sin\alpha,\ p\sin\alpha + p'\cos\alpha),$$

where the dash stands for $d/d\alpha$. Show also that the radius of curvature of C at P is $\pm(p + p'')$.

If C is a simple closed curve containing O in its interior and everywhere convex (i.e. lying entirely to one side of each of its tangents), prove that the perimeter of C is of length

$$\int_0^{2\pi} p d\alpha,$$

and encloses an area

$$\frac{1}{2}\int_0^{2\pi}(p^2 - p'^2)d\alpha.$$

Question (1942 STEP I Q108)

Sketch, in the same figure, the curves whose equations in polar coordinates are:

1. $r = a + b\cos\theta$; (ii) $r = b + a\cos\theta$;

where $a > b > 0$. Prove that, when the acute angle between the tangent to (i) at P and the radius vector OP from the origin O has its least value, OP is a tangent to (ii) at O, and that OP is then equally inclined to the initial line and to the tangent to (i) at P.

Question (1918 STEP I Q112)

Trace the curve

$$r^3 \sin 4\theta = \sin(\theta + \alpha)$$

(a) when $0 < \alpha < \frac{1}{4}\pi$ and (b) when $\alpha = 0$. Shew how the shape of the curve passes over into its limiting form as α tends to zero.

Question (1925 STEP I Q111)

A curve C' is obtained by inverting the spiral $r = ae^{m\theta}$ with respect to the circle with centre $(-a, 0)$ and radius $a\sqrt{2}$. Shew that, if s' is the arc of C' measured from the origin to the point P' which corresponds to the point 'θ' of the spiral, then

$$\frac{ds'}{d\theta} = \frac{a\sqrt{1+m^2}}{\cosh m\theta + \cos\theta}.$$

Prove that C' is symmetrical about the origin, and draw a rough sketch of the curve.

Question (1927 STEP I Q112)

A point P has polar co-ordinates connected by the relation

$$\theta = \int \frac{\sqrt{a(1-e^2)}dr/r}{\sqrt{a(1-e)\sqrt{-a(1-e^2)+2r-r^2/a}}},$$

where $a > 0, e^2 < 1$. Shew that P lies on an ellipse having a focus at the origin and a, e for its semi-major axis and eccentricity respectively. Shew further that

$$\int \frac{rdr}{\sqrt{a(1-e)\sqrt{-a^2(1-e^2)+2a^3r-a^2r^2}}} = \phi - e\sin\phi,$$

where ϕ is the eccentric angle of the point (r, θ) of the ellipse.

Question (1929 STEP I Q110)

(a) Show that if two curves are polar reciprocals in the circle $r = a$ their radii of curvature at corresponding points are connected by the relation $\rho_1\rho_2 = \frac{r_1^3 r_2^3}{a^4}$, where r_1, r_2 are the distances from the pole to the corresponding points, and p_1, p_2 the radii of curvature. (b) If two curves are inverses in $r = a$, show that $r\frac{d^2p}{dr^2} - \frac{dp}{dr}$ has the same value at corresponding points on each of them.

Question (1920 STEP I Q106)

Obtain the equation of a conic in polar coordinates, the focus being the pole, in the form

$$r(1 + e\cos\theta) = l,$$

and shew that, if $2l = e^{-1} - e$, the equation represents for different values of e a family of confocal conics, whose foci are at unit distance apart. Prove also that if in this special equation r and θ are replaced by r^2 and 2θ, the curves represented are again a family of confocal conics, the distance between the foci now being two units of length.

Question (1917 STEP I Q116)

Trace the curve $r = a(2\cos\theta - 1)$. Find the areas of the loops and shew that their sum is $3\pi a^2$.

Question (1941 STEP I Q106)

Trace the curve

$$x^5 + y^5 = 5ax^2y^2 \quad (a > 0).$$

By writing $y = tx$, or otherwise, prove that the area of the loop is $\frac{5}{2}a^2$.

Question (1913 STEP II Q209)

Find the equation of the tangent at any point of the curve $x = f(t), y = F(t)$. If Y is the foot of the perpendicular from the origin O on the tangent to the curve $ay^2 = x^3$ at any point $P(at^2, at^3)$, show that the coordinates of Y are $\dfrac{3at^4}{9t^2+4}$ and $\dfrac{2at^3}{9t^2+1}$. Deduce that, if $OY = p$ and the inclination of OY to the axis of x is ψ,

$$27p = 4a\cos\psi\cot^2\psi.$$

Question (1915 STEP II Q210)

Prove the formulae for the radius of curvature of a curve

$$\rho = \frac{rdr}{dp} = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$$

Any point on a curve is taken as pole and the tangent at it is the initial line, prove that the approximate equation of the curve in the neighbourhood of the origin is $r = 2\rho\theta + \frac{4}{3}\rho\frac{d\rho}{ds}\theta^2$, where ρ and $\frac{d\rho}{ds}$ are the values at the origin.

Question (1921 STEP II Q210)

Trace the curve $r = a(1 + 2\cos\theta)$, and show in the figure the area represented by

$$\frac{1}{2}\int_0^{2\pi} r^2\, d\theta.$$

Find separately the areas bounded by each loop of the curve.

Question (1923 STEP II Q210)

Find an expression for the area of a closed curve in terms of polar coordinates. Show that the area enclosed by the curve

$$r = a\frac{\cos\theta + 3\sin\theta}{(\cos\theta + 2\sin\theta)^2}$$

and the two radii from the origin for $\theta = 0$ and $\theta = \frac{\pi}{4}$ is $37a^2/162$.

Question (1924 STEP II Q210)

Sketch the curve $a^2y^2 = x^2(a^2 - x^2)$. Find the area of a loop of the curve, and prove that the volume generated by revolution of a loop about the y-axis is $\pi a^3/4$.

Question (1927 STEP II Q210)

Give a rough sketch of the curve

$$3x^2 = y(y-1)^2,$$

and determine the greatest breadth of the loop, its perimeter and its area.

Question (1942 STEP II Q207)

Sketch the curve

$$xy = x^3 + y^3,$$

and find (i) the radii of curvature at the origin of coordinates, (ii) the area of the loop.

Question (1914 STEP IIII Q202)

Make a sketch, correct in its essential details, showing the orthogonal projections of the meridians and parallels of a terrestrial globe on to the tangent plane at a place in latitude 30° N. Compare the geometry of the figure in any respects with that of an ellipse and the circles having double contact with it.

Question (1919 STEP I Q309)

Prove that if (r, θ) are the polar coordinates of a point on a curve and p is the length of the perpendicular from the origin on the tangent at the point, the radius of curvature is given by

$$\rho = r\frac{dr}{dp}.$$

Shew that the radius of curvature at any point on a conic is $2a\text{cosec}^3\phi$, where $4a$ is the latus rectum and ϕ the angle which the tangent at the point makes with a focal distance.

Question (1920 STEP I Q310)

Define the polar plane of a point with regard to a sphere; and shew that if points are taken on a straight line their polar planes pass through another straight line.

Question (1942 STEP I Q309)

A plane curve is referred to polar coordinates r, θ. The perpendicular from the origin upon the tangent to the curve is p. If $u = 1/r$, prove that

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2,$$

and that ρ, the radius of curvature, is given by

$$\frac{1}{\rho} = u^3 \frac{u + \frac{d^2u}{d\theta^2}}{\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\}^{\frac{3}{2}}}.$$

Question (1915 STEP II Q308)

Find the area of a loop of the curve $y^2 = x^2 - x^4$.

Find also the distance from the origin of the centre of gravity of the area included within the loop.

Question (1916 STEP II Q309)

Trace the curve $r = a(\sin\theta - \cos 2\theta)$, and find the area of the loop which passes through the point $(2a, \frac{\pi}{2})$.

Question (1918 STEP II Q308)

Prove the formulae $\rho = r\frac{dr}{dp}$ and $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$. Find the radius of curvature at any point of the cardioid $r = a(1 + \cos\theta)$.

Question (1925 STEP II Q310)

Trace $r = a(2\cos\theta - 1)$, find the areas of its loops and show that their sum is $3\pi a^2$.

Question (1927 STEP II Q306)

Find in polar coordinates an expression for the angle between the radius vector to a point on a curve and the tangent at that point. Prove that the locus of the centre of a circle passing through the pole and touching the curve $r^m = a^m \cos m\theta$ is the curve $(2r)^n = a^n \cos n\theta$, where $n(1-m) = m$.

Question (1936 STEP III Q305)

Taking $(1/u, \theta)$ as the polar coordinates of a point of a plane curve, obtain an expression for the curvature in terms of u, $\frac{du}{d\theta}$, and $\frac{d^2u}{d\theta^2}$. Shew that the curvature of the curve $au = \cosh n\theta$ has a stationary value provided $3n^2$ is not less than 1. Determine whether this value is a maximum or minimum.

Question (1914 STEP IIII Q310)

Shew how to find the area of a closed curve, whose equation in polar coordinates is given. Find the area of a loop of the curve $r = a\cos^2 n\theta$, where n is a positive integer.

Question (1920 STEP IIII Q310)

Trace the curve $r = a(\cos\theta + \cos 2\theta)$, and shew that the curve crosses itself at the points $(a/\sqrt{2}, \pm\frac{1}{4}\pi)$. Prove that the area of that portion of the largest loop that is not common to the other loops is $\sqrt{2}a^2$.

Question (1934 STEP I Q409)

Prove that in polar coordinates (r, θ) the radius of curvature of a curve is given by

$$\frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$$

Prove that the radius of curvature of the curve $r = a(1-\cos\theta)$ is $\frac{4a}{3}\sin\frac{\theta}{2}$. Sketch the curve.

Question (1920 STEP II Q408)

Prove that if ϕ is the angle between the radius vector and the tangent at any point of a curve $\tan\phi = r\frac{d\theta}{dr}$. Find ϕ at the point (r, θ) on the curve $r^2 = a^2\cos 2\theta$ and prove that if p is the perpendicular from the origin on the tangent at the point $pa^2 = r^3$. Find also the equation of the first positive pedal.

Question (1921 STEP II Q407)

Prove that if p is the perpendicular from the origin on the tangent to a curve $r = f(\theta)$,

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

Prove that the feet of the perpendiculars from the origin on the normals to the curve $r^2 = a^2 \cos 2\theta$ lie on the curve

$$4r^2/a^2 = \cos 2\theta + \cos(\frac{1}{3}\pi + \theta).$$

(Note: The constant in the cosine term is transcribed as it appears.)

Question (1923 STEP II Q408)

If ϕ is the angle between the radius vector and the tangent of a curve, prove that

$$\tan\phi = r\frac{d\theta}{dr}.$$

Find ϕ at a point of the curve $r^2 = a^2 \cos 2\theta$. If the normal at the point (r, θ) cuts the initial line in G, prove that $OG = 2ar(a^2 + r^2)/(a^4 + 2r^4)$ where O is the origin.

Question (1915 STEP III Q410)

Find the area of the surface generated by the revolution of the lemniscate $r^2 = a^2 \cos 2\theta$ round the initial line.

Question (1916 STEP III Q406)

Prove the formula for radius of curvature $\rho = r\frac{dr}{dp}$. In the curve $r^n = a^n \cos n\theta$ the angle between the radius vector to (r, θ) and the radius to the centre of curvature at that point is ψ. Prove that $n \tan\psi = \tan n\theta$ and that the distance of the centre of curvature from the origin is $nr \sec\psi/(n + 1)$.

Question (1917 STEP III Q406)

Prove the formula for the radius of curvature at any point of a curve, using polar co-ordinates. Find the radius of curvature at any point of the curve $r^2 = a^2 \cos 2\theta$ and prove that its evolute is

$$9(x^{4/3} + y^{4/3})(x^{2/3} - y^{2/3}) = 4a^2.$$

Question (1926 STEP III Q409)

C is the centre and P a given point ($CP = b$) on a spoke of a wheel of radius a that rolls along a straight line on the horizontal ground, all the motion being in a vertical plane. For the trochoidal locus of P, prove that the curvature is numerically equal to $\frac{b}{(a-b)^2}$ at the highest point, $\frac{b}{(a+b)^2}$ at the lowest point, and zero when APC is a right angle, where A is the point of contact of the rolling circle with the ground.

Question (1937 STEP III Q407)

Determine the surface area and volume of the solid figure obtained by revolving the curve $r = a(1 + 2\cos\theta)$ about its axis of symmetry.

Question (1938 STEP III Q405)

Establish the result for the radius of curvature at any point of a plane curve whose tangential-polar (p, ψ) equation is given. If in such a curve the intercept on any tangent between the point of contact and the foot of the perpendicular from the origin on to the tangent is $p + a$, where a is a constant, find the angle between the tangents at the points for which $p = a$ and $p = 2a$ respectively. Find the radius of curvature at the first of these points.

Question (1919 STEP IIII Q408)

Prove that in polar coordinates $r\frac{d\theta}{dr}$ is the tangent of the angle between the radius vector and tangent to a curve. In the case of the curve $r^n = a^n \cos n\theta$, prove that $a^n \frac{d^2 r}{ds^2} + nr^{2n-1} = 0$.

Question (1931 STEP IIII Q404)

Trace the curve

$$x = 2a\sin^2 t\cos 2t, \quad y = 2a\sin^2 t\sin 2t.$$

Show that the length of its arc is $8a$, and find the radius of curvature at the origin.

Question (1933 STEP IIII Q403)

Sketch the curve

$$x^3 = 3xy^2 + a^2x + y^2.$$

Trace the inverse of the curve in the circle

$$x^2 + y^2 = 1,$$

and find the area of a loop of this inverse.

Question (1917 STEP II Q504)

Draw the graphs of cosech x and $\sinh\frac{1}{x}$, and determine on which side of the hyperbola $xy = 1$ each curve lies.

Question (1918 STEP II Q508)

Prove the expressions for the radius of curvature of a curve

$$\text{(i) } \rho = r\frac{dr}{dp}, \quad \text{(ii) } \rho = p + \frac{d^2p}{d\psi^2}.$$

Find ρ and p at a point of $r^n = a^n \sin n\theta$.

Question (1918 STEP II Q510)

Shew how to find the area of a curve given in polar coordinates. Trace the curve $r = a(2\cos\theta + \sqrt{3})$ and find the area between the two loops.

Question (1931 STEP III Q506)

Prove the formula $\rho = r\frac{dr}{dp}$ for the radius of curvature of a curve at a point P where the length of the radius vector is r and the length of the perpendicular from the origin on to the tangent at P is p. Sketch roughly the curve for which $\frac{p}{r}$ is a constant and shew that ρ also is in constant ratio to r and p. Prove that the radius of curvature can never be less than the corresponding radius vector.

Question (1932 STEP III Q508)

Sketch the curve whose polar equation is $r^2(\sec n\theta + \tan n\theta) = a^2$, where n is a positive integer and a is a constant. In the case $n = 1$ shew that the only real point at which the circle of curvature passes through the pole is given by $\theta = \tan^{-1}\sqrt{1 + \sqrt{\frac{28}{3}}}$.

Question (1933 STEP III Q507)

Trace the curve $r\cos\theta + a\cos 2\theta = 0$. Shew that the area of the loop is $a^2(2 - \frac{\pi}{2})$, and that the area enclosed between the curve and its asymptote is $a^2(2 + \frac{\pi}{2})$.

Question (1933 STEP III Q509)

Shew that the area of the surface of the prolate spheroid obtained by the rotation of an ellipse of eccentricity e about its major axis $(2a)$ is

$$A = 2\pi a^2 \left[\sqrt{1-e^2} + \frac{\sin^{-1} e}{e}\right]$$

and that the centroid of the half surface bounded by the central circular section is at a distance d from the plane of that section, where

$$Ad = \frac{4\pi a^3}{3}\frac{1}{e^2}\left[\sqrt{1-e^2} - (1-e^2)^{\frac{3}{2}}\right].$$

Question (1917 STEP IIII Q508)

Prove the formulae for the radius of curvature ρ of a curve

$$\text{(i) } r\frac{dr}{dp}, \quad \text{(ii) } \frac{\{r^2+\left(\frac{dr}{d\theta}\right)^2\}^{\frac{3}{2}}}{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}.$$

If n is the length of the normal at a point of the curve $r=a(1+\cos\theta)$ intercepted between the curve and the initial line, prove that

$$4n-3\rho : 2n = a : r.$$

Question (1916 STEP IIIII Q510)

Find the area between the curve

$$y^2(3a-x)=x^3$$

and its asymptote.

Question (1926 STEP II Q612)

Sketch the curve given by the equation

$$y^2=\frac{x^2(3a-x)}{a+x}.$$

Shew that the coordinates of any point on the curve may be taken as $(a\sin 3\theta\csc\theta, a\sin 3\theta\sec\theta)$, and prove that the area of the loop of the curve is equal to the area between the curve and its asymptote.

Question (1916 STEP III Q610)

For a curve defined by $p=f(\psi)$, prove that the projection of the radius vector on the tangent is $\frac{dp}{d\psi}$ and that $\rho=p+\frac{d^2p}{d\psi^2}$. For the curve $p=a\sin 2\psi$, prove that $r^2=a^2-3p^2$ and that the p,r equation of the locus of centres of curvature is $r^2=16a^2-3p^2$.

Question (1917 STEP III Q610)

Prove that for a plane curve the radius of curvature $\rho=r\frac{dr}{dp}$. Shew that the radius of curvature at a point of the curve $r^n=a^n\cos n\theta$ is $a^n/\{(n+1)r^{n-1}\}$.

Question (1920 STEP III Q602)

Sketch the curve $r(\cos\theta+\sin\theta)=a\sin 2\theta$, and find the area of the loop of the curve.

Question (1920 STEP III Q611)

If n is the length of the normal intercepted between a point (r, θ) of the curve

$$r^2 = a^2 \cos 2\theta$$

and the initial line, and ρ is the radius of curvature at the point, prove that

$$n(2r + 3\rho) = 3r\rho.$$

Question (1925 STEP III Q610)

Prove that, if P and Q are points on the cardioid $r = a(1 + \cos\theta)$ such that the angle between the tangents at P and $Q = \alpha$, the chord PQ subtends an angle $\frac{1}{2}(\pi - \alpha)$ at the cusp.

Question (1918 STEP II Q714)

Prove that the area of one loop of the curve $x^4 - 2xya^2 + a^2y^2 = 0$ is $\frac{1}{6}a^2$.

Question (1919 STEP II Q707)

If ϕ is the angle between the radius vector and the tangent to the curve $f(r, \theta) = 0$, prove that $\tan\phi = r\frac{d\theta}{dr}$. Prove that, if the tangents at P, Q, two points on the curve $r = a(1 - \cos\theta)$, are parallel, the chord PQ subtends an angle $2\pi/3$ at the pole.